LECTURE 4: COMPLETENESS (II) AND LIMITS (I)

Today: Two very nice applications of sup, starting with the Archimedean Property:

1. Archimedean Property

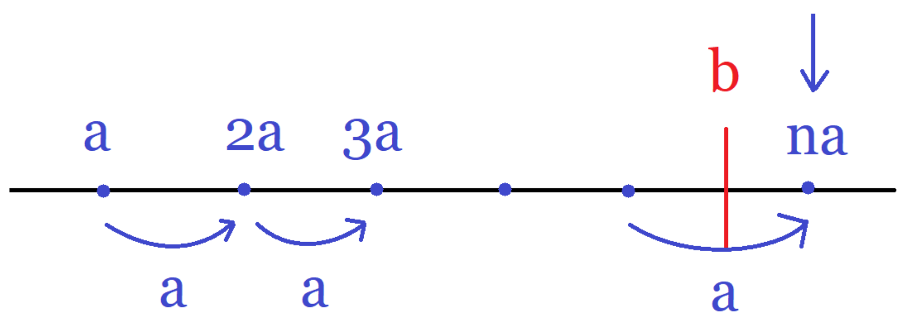
Video: Archimedean Property

Analogy: Suppose you go to the grocery store and the cashier says "Your total is \$100" Can you pay this using (infinitely many) \$1 bills? What if the total is \$1000 and you only have 1 cent coins? Still yes! This is the essence of:

Archimedean Property

If a and b are positive are real numbers, then for some $n \in \mathbb{N}$ we have $na > b$

Interpretation: No matter how large the total b is and how small your currency a is, it is always possible to exceed b by using enough a

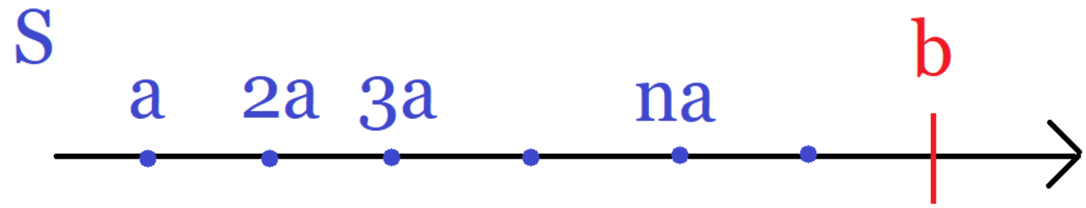


Date: Thursday, September 9, 2021.

Or, to quote the book, "Given enough time, one can empty a large bathtub with a small spoon."

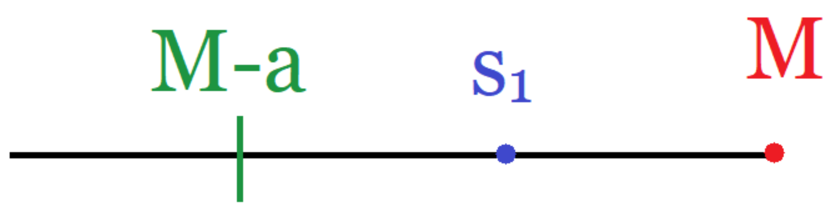
Proof: Assume $a < b$ (the other cases are boring[1])

Suppose this is false, that is there are $a > 0$ and $b > 0$ such that $na \leq b$ for all $n \in \mathbb{N}$:



This means that if you let $S = \{na \,|\, n \in \mathbb{N} \cdots\}$, then S is bounded above by b, so by the **Least Upper Bound Property**, $M =: \sup(S)$ exists.

Consider $\boxed{\text{M-a}} < M$, then by the definition of $\sup(S)$, there is $s_1 \in S$ such that $s_1 > M - a$



Since $s_1 \in S$, $s_1 = n_0 a$ for some $n_0 \in \mathbb{N}$. Therefore:

$$s_1 > M - a \Rightarrow n_0 a > M - a \Rightarrow n_0 a + a > M \Rightarrow (n_0 + 1)a > M$$

[1]If $a > b$, $n = 1$ works, and if $a = b$, $n = 2$ works

This is a contradiction because, since $(n_0+1)a \in S$ (by definition) and M is an upper bound, we must have

$$M < (n_0 + 1)a \leq M \Rightarrow\Leftarrow \quad \square$$

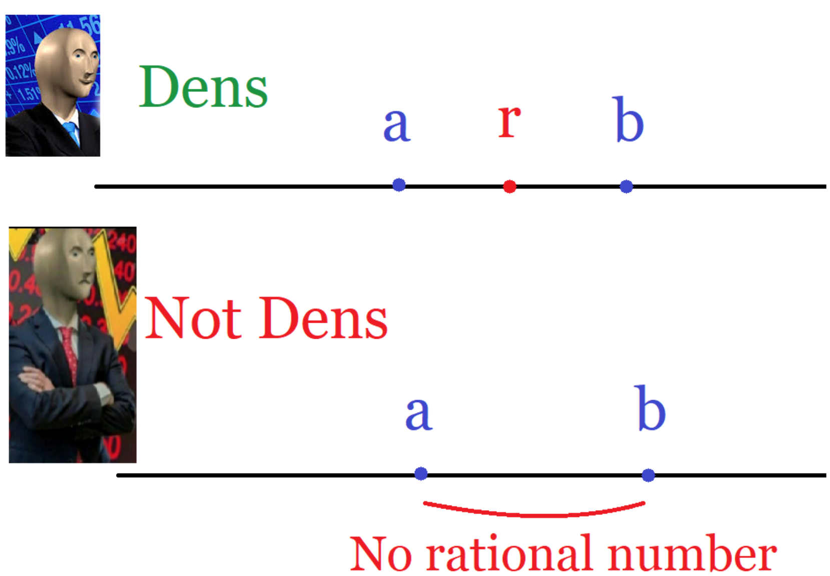
2. Dense with me!

Video: $\mathbb{Q}$ is dense in $\mathbb{R}$

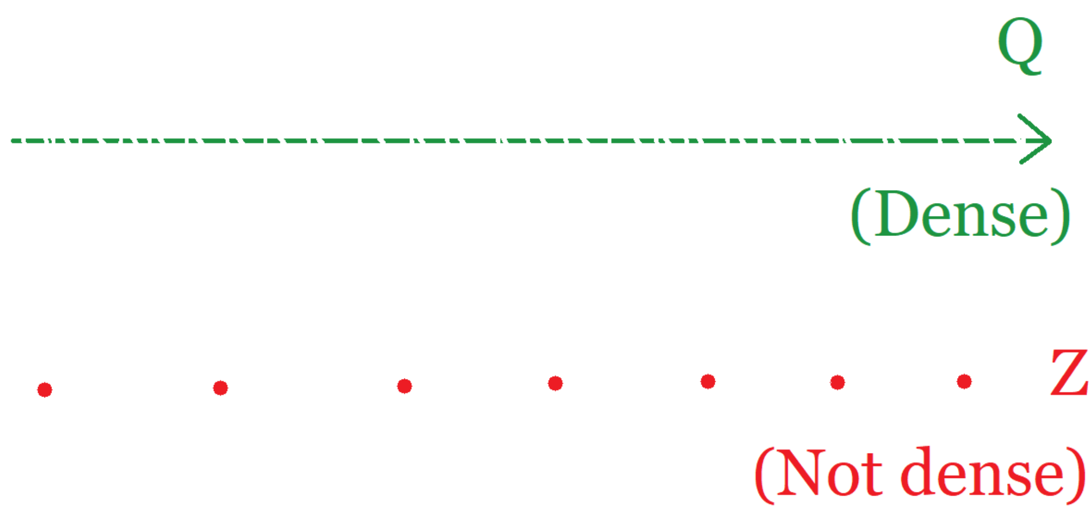
Finally, using the Archimedean property, we can show the following **important** fact about $\mathbb{Q}$, It says that between two rational numbers there always is a real number:

Theorem ($\mathbb{Q}$ is dense in $\mathbb{R}$)

For any real numbers a and b with $a < b$ there is a rational number r such that $a < r < b$

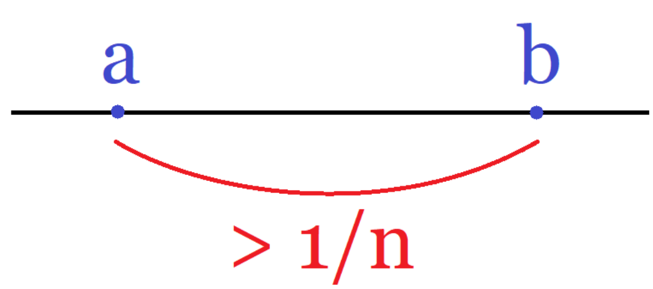


Note: The point is that even if a and b are really close together, you can always squeeze a rational number between them. Intuitively this is saying that, even though $\mathbb{Q}$ has holes, it still fills up "most" of $\mathbb{R}$, unlike $\mathbb{Z}$ for instance, which is pretty sparse.



Proof: Suppose $a < b$, WTF $r = \frac{m}{n}$ such that $a < \frac{m}{n} < b$.

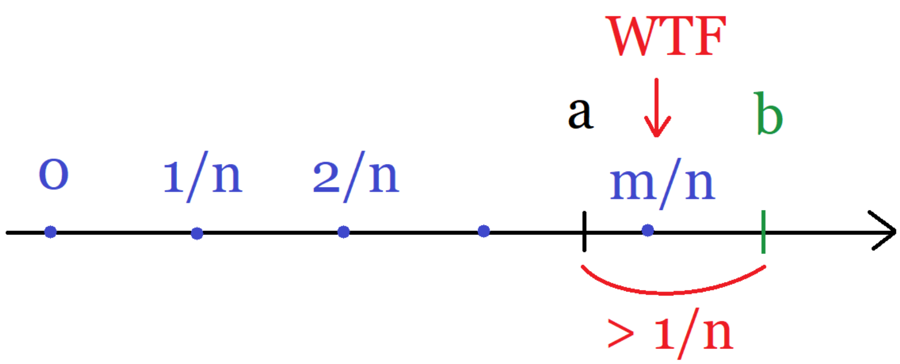
STEP 1: Since $b - a > 0$, by the Archimedean property applied to $b - a$ (your currency) and 1 (your total), there is $\boxed{n \in \mathbb{N}}$ such that $n(b-a) > 1$, that is $b - a > \frac{1}{n}$



Note: For the remainder of the proof, remember that n is **fixed** (if you want, think $n = 3$). Think of $\frac{1}{n}$ as a scale, like 1 mm or 1 nm.

STEP 2: WLOG, assume[2] $a > 0$

Main Idea: List all the fractions $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots$ until you reach the last one that is $< b$. This process has to stop because b is finite, and also that last fraction is guaranteed to be between a and b because a and b are at least $\frac{1}{n}$ apart.



Here are the details: Consider the following set:

$$S = \left\{ \frac{m}{n} \mid m = 0, 1, 2, \cdots \text{ and } \frac{m}{n} < b \right\}$$

Then S is nonempty ($0 \in S$) and S is bounded above by b, so by the Least Upper Bound Property, $\sup(S) = r$ exists.

Claim: This r solves our problem

[2]For the other cases: If $a < b < 0$, then notice $-a > -b > 0$ and use this proof to find r rational with $-b < r < -a$ and then $-r$ does the trick. And if $a = 0$ then $r = \frac{1}{n}$ works since $b - a > \frac{1}{n}$ and if $b = 0$ then $r = -\frac{1}{n}$ works. And if $a < 0 < b$, then let $r = 0$

Why? Need to show that r is rational, and $a < r < b$.

First of all, S only has finitely many elements: By the Archimedean property with $\frac{1}{n}$ (currency) and b (total), there is $k \in \mathbb{N}$ such that $k\left(\frac{1}{n}\right) = \frac{k}{n} > b$, so S has at most k elements.

Since S is finite, $\sup(S) = \max(S)$ (Think for example $S = \{1, 3, 5, 9\}$. You can compare all the elements of S one by one and pick the one that is is largest[3]). In particular, $r =: \sup(S) = \max(S) \in S$ (by definition of max), so by definition of S, r is rational and $r < b$.

Finally, to show $r > a$, suppose $r = \frac{m}{n} \leq a$, but then

$$b - a > \frac{1}{n} \Rightarrow b > a + \frac{1}{n} \geq \frac{m}{n} + \frac{1}{n} = \frac{m+1}{n}$$

Hence $\frac{m+1}{n} < b$ and so $\frac{m+1}{n}$ is an element of S that is **bigger** than $r = \frac{m}{n}$, which contradicts the fact that $r = \sup(S) \Rightarrow\Leftarrow$ □

3. Interlude: What is ∞?

Video: What is Infinity?

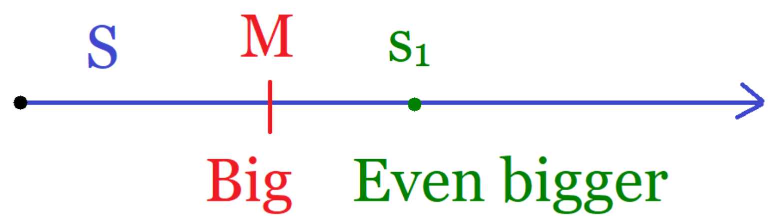
As an interlude, let's digress and talk about infinity. The nice thing is that we can extend the definition of $\sup(S)$ and $\inf(S)$ in the case where S is unbounded:

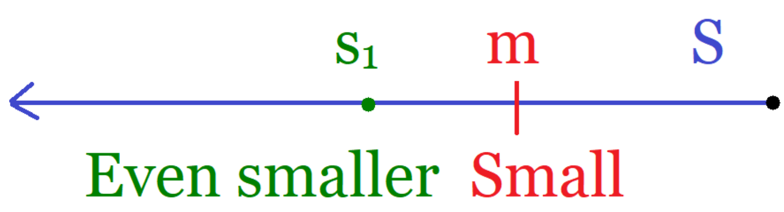
[3]To make this rigorous, you can use induction on the size of S. Namely P_n would be the proposition "If S has n elements, then S has a max"

Definition:

We say $\sup(S) = \infty$ if S is not bounded above, that is: for all M there is $s_1 \in S$ such that $s_1 > M$.

Similarly, $\inf(S) = -\infty$ if S is not bounded below, that is: for all m there is $s_1 \in S$ such that $s_1 < m$.





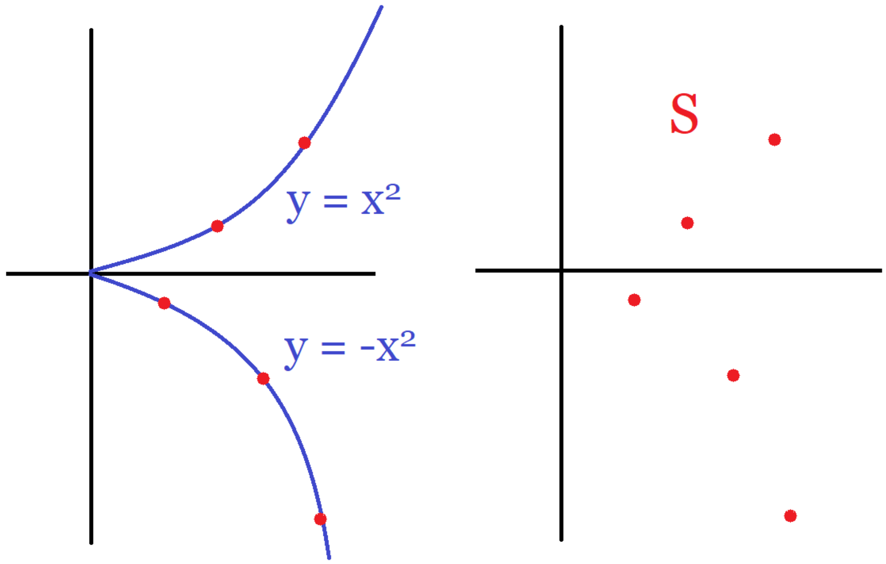
With this new definition, we get that $\sup(S)$ and $\inf(S)$ always exist (but could be $\pm\infty$)

Example:

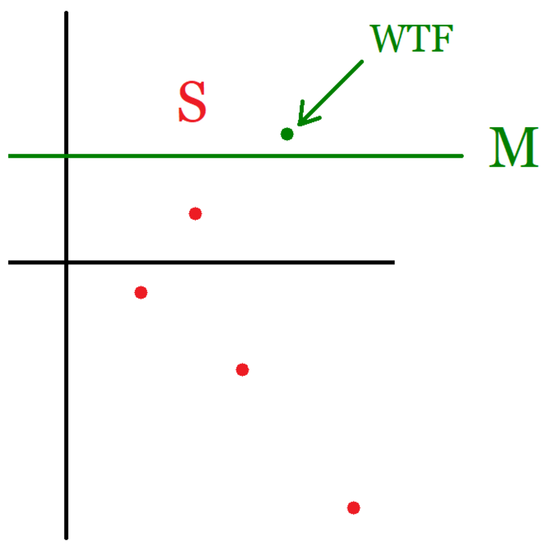
Find $\sup(S)$ where

$$S = \left\{n^2(-1)^n \mid n \in \mathbb{N}\right\} = \{-1, 4, -9, 16, \cdots\}$$

Picture: Notice $n^2(-1)^n$ jumps back and forth between $y = x^2$ and $y = -x^2$



Let M be given, we need to find some $s_1 = n^2(-1)^n \in S$ such that $s_1 > M$.



Scratchwork: Notice that $n^2(-1)^n$ gets bigger for *even* n, and if n is even, then $n^2(-1)^n = n^2$. Also $n^2 > M \Rightarrow n > \sqrt{M}$

Actual Proof: Let n be any **even** integer[4] such that $n > \sqrt{M}$ and let $s_1 = n^2(-1)^n \in S$, then

$$s_1 = n^2(-1)^n = n^2 > (\sqrt{M})^2 = M\checkmark$$

Hence $\sup(S) = \infty$.

Note: This officially concludes our exploration of the real numbers. If you're interested, at the end there is an **optional** discussion of how to construct the real numbers and actually prove the least upper bound property, which I invite you to check out.

4. Sequences

Video: What is a Sequence?

With that said, welcome to our Sequence adventure! In this chapter, we'll study sequences, which are infinite lists of numbers.

Intuitively:

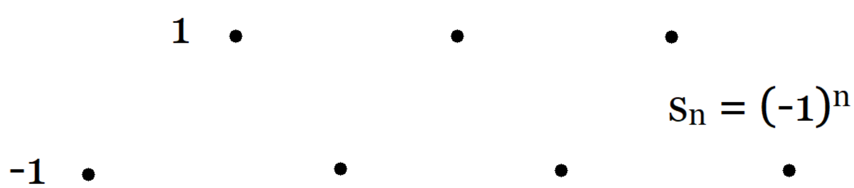
A sequence $(s_n)_{n\in\mathbb{N}}$ is an infinite list of real numbers.

Examples:

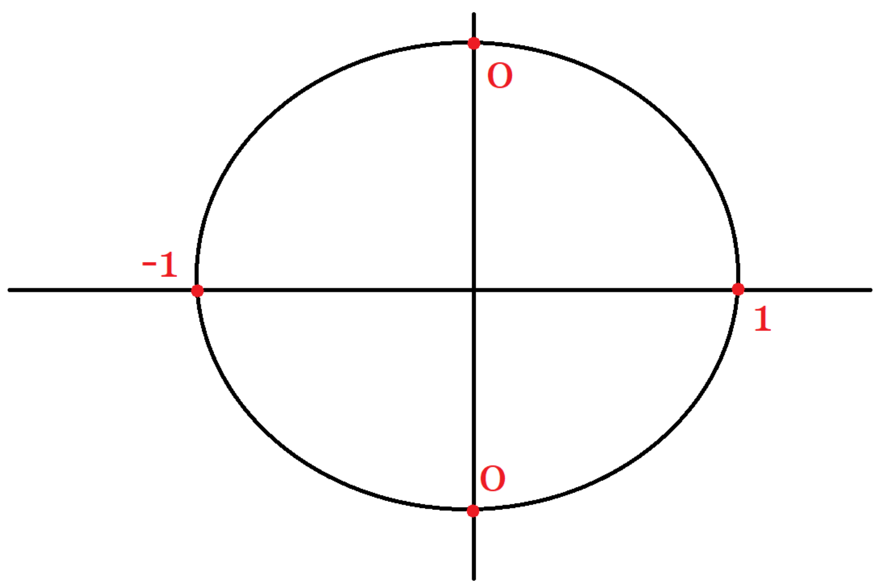
(1) $s_n = \frac{1}{n^2}, n \in \mathbb{N}$, so $(s_n) = \left(1, \frac{1}{4}, \frac{1}{9}, \cdots\right)$

[4]In the case $M < 0$, just let $n = 2$

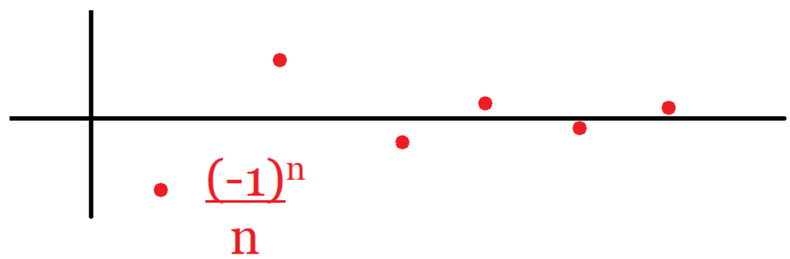
(2) $s_n = (-1)^n, n \geq 0$, so $(s_n) = (1, -1, 1, -1, 1, \cdots)$. This sequence jumps back and forth between 1 and -1. It's a *great* source of counterexamples.



(3) $s_n = \cos\left(\frac{\pi n}{2}\right), n \geq 0$, which is $(1, 0, -1, 0, 1, 0, -1, 0, \cdots)$



(4) $s_n = \frac{(-1)^n}{n}$, which jumps back and forth between positive and negative values, but which seems to go to 0



Actual Definition:

A sequence $(s_n)_{n\in\mathbb{N}}$ is a function from $\mathbb{N}$ to $\mathbb{R}$

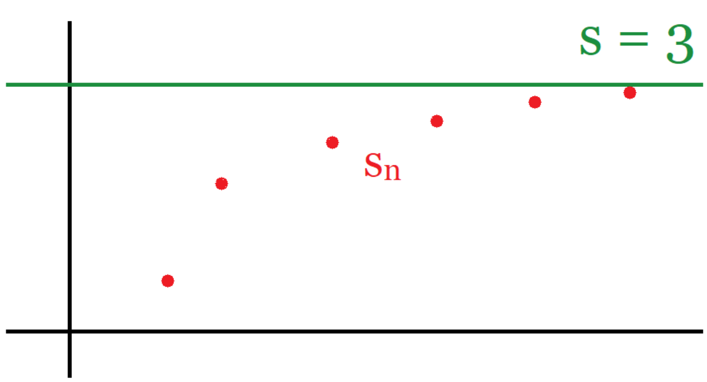
Why? Because for natural number n, you associate a real number s_n. For example, the sequence $s_n = \frac{1}{n^2}$ is the same as the function $f(n) = \frac{1}{n^2}$. In fact $f(1) = 1, f(2) = \frac{1}{4}, f(3) = \frac{1}{9} \cdots$.

5. Limits of Sequences

Video: What is a limit?

Goal: Figure out what happens to s_n as n goes to ∞.

Example: Consider $s_n = 3 - \frac{1}{n^2}$.

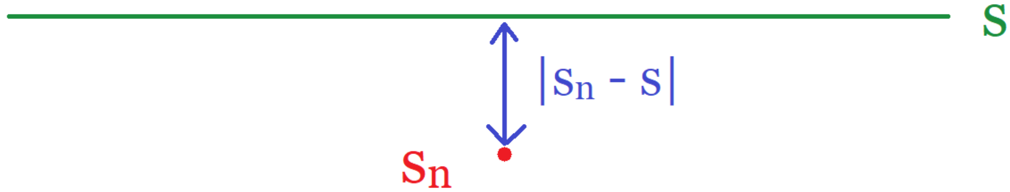


Intuitively, s_n approaches to $s = 3$ as n goes to ∞, and our goal is to make this rigorous.

Intuitively:

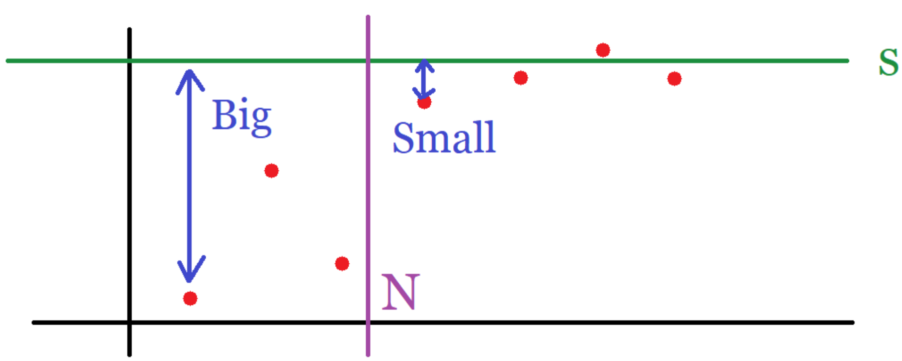
$\lim_{n\to\infty} s_n = s$ means that if n is large, then s_n goes to s

First of all, s_n goes to s means that $\text{dist}(s_n, s) = |s_n - s|$ is small.



In other words, we can make $|s_n - s|$ as small as we want, by letting n be large enough.

That is, there is some threshold N such that, after N, $|s_n - s|$ is as small as we want.



Finally, what does as small as we want mean? For astronomers, 10 km is small, but 10 km for microbiologists is actually pretty big! In some sense, we want a definition that makes everyone happy.

Let $\epsilon > 0$ (think error or tolerance, so $\epsilon = 10$ km for astronomers, and $\epsilon = 1$nm for biologists), then:

Ultra Important Definition of a Limit:

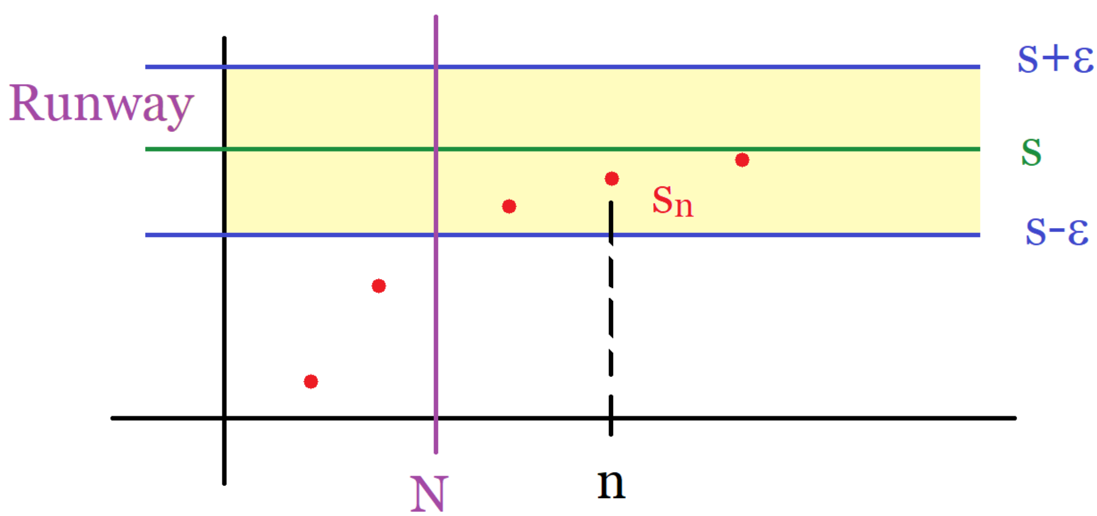
$$\lim_{n\to\infty} s_n = s \qquad \text{means:}$$

For all $\epsilon > 0$ there is some N such that if $n > N$, then $|s_n - s| < \epsilon$

That is, no matter how small our error ϵ is, there is some threshold N such that, once you pass the threshold ($n > N$) then your can guarantee that s_n is at most ϵ away from s.

Note:

$$|s_n - s| < \epsilon \Rightarrow -\epsilon < s_n - s < \epsilon \Rightarrow s - \epsilon < s_n < s + \epsilon$$



So in other words we can always guarantee that s_n is in the small interval $(s - \epsilon, s + \epsilon)$

Analogy: Think of s_n as an airplane and $(s - \epsilon, s + \epsilon)$ as a runway. What this is saying is that, no matter how small our runway is, we can always guarantee that the plane s_n lands in the runway if N is large enough.

In the next lectures, we'll practice with the rigorous definition of a limit, so that you can get a feel for it. This is very important for the exams.

6. Optional: Construction of $\mathbb{R}$

Video: Construction of $\mathbb{R}$

Even though we've been talking about the real numbers, we never actually defined what a real number is! That's precisely what we're going to do now. As an added benefit, we'll be able to *prove* the least upper bound property (so it's not an axiom after all!)

Goal: Construct the real numbers from the rational numbers.

Motivation: How would you define $\sqrt{2}$ using only rational numbers, without ever mentioning the number $\sqrt{2}$?

Consider the following set S (recall $\sqrt{2} \approx 1.414$)

$$S = \left\{ r \in \mathbb{Q} \mid r < \sqrt{2} \right\} = \left\{ 1, -2, \frac{4}{7}, 0, 1.2, 1.41, -3.6, \cdots \right\}$$

In other words, S is the set of all rational numbers that come before $\sqrt{2}$.

Upshot: Strictly speaking, the set $\left\{1, -2, \frac{4}{7}, 0, 1.2, 1.41, -3.6, \cdots\right\}$ doesn't mention $\sqrt{2}$ at all; to the naked eye, it is just a random set of rational numbers. And that's precisely how we'll define real numbers, simply as *special* sets of rational numbers.

Definition:

A real number S is a subset of $\mathbb{Q}$ with the following properties[a] (called a Cut):

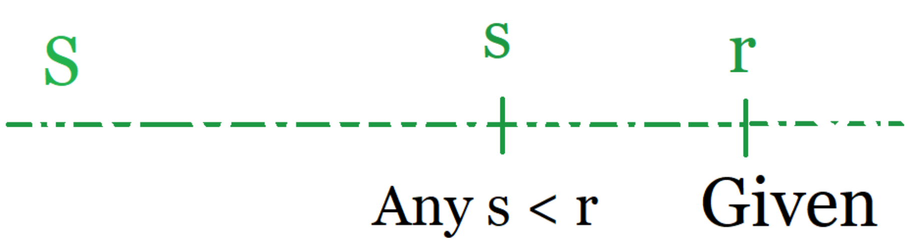
(1) $S \neq \emptyset$ and $S \neq \mathbb{Q}$

(2) If $r \in S$ and if s is any rational with $s < r$, then $s \in S$

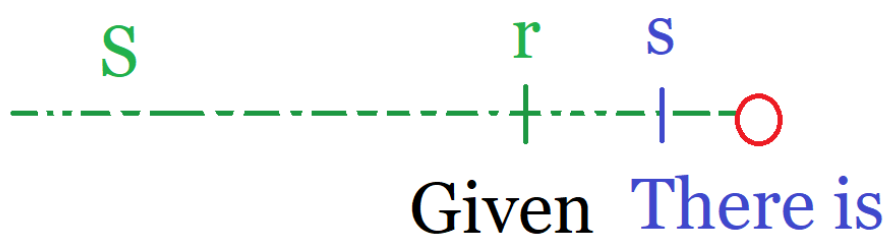
(3) If $r \in S$, then there is some $s \in S$ such that $s > r$

[a] The book uses α instead of S

Note: (2) is just saying that S contains all the rationals before r, like in the following picture:



Note: (3) is saying that S has no maximum: No matter which $s \in S$ you pick, there's always a bigger element $r \in S$



Notice the subtle difference between (2) and (3). In (2) we say **for all** $s < r$, $s \in S$, but in (3) we say **there is** $s \in S$ with $s > r$

Important Example: If $a \in \mathbb{Q}$ is given, then

$$S = a^{\star} = \{r \in \mathbb{Q} \mid r < a\}$$

This is a rational cut and written as $a^{\star}$,

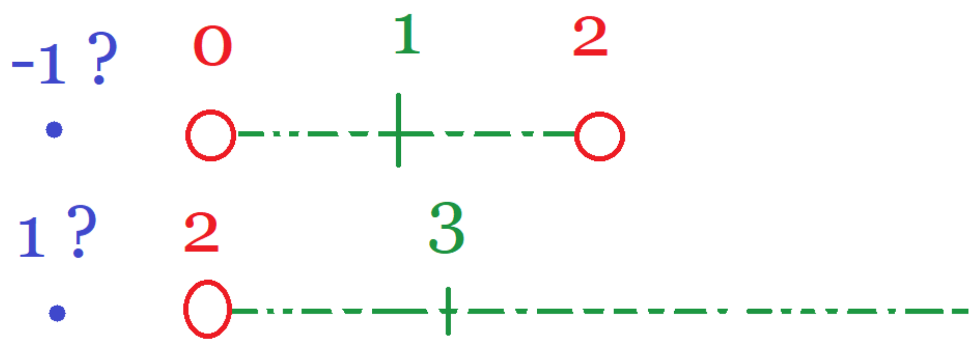
So for instance

$$1^{\star} = \{r \in \mathbb{Q} \mid r < 1\} = \{-2.3, -1, 0, 0.5, 0.99, \cdots\}$$

Non-Examples:

(a) $\emptyset, \mathbb{Q}$: doesn't satisfy (1)

(b) $S = \{r \in \mathbb{Q} \mid 0 < r < 2\}$: Doesn't satisfy (2): $1 \in S$ and $-1 < 1$ but $-1 \notin S$. Similarly, $S = \{r \in \mathbb{Q} \mid r > 2\}$ is not a cut: $3 \in S$ and $-1 < 3$ but $-1 \notin S$



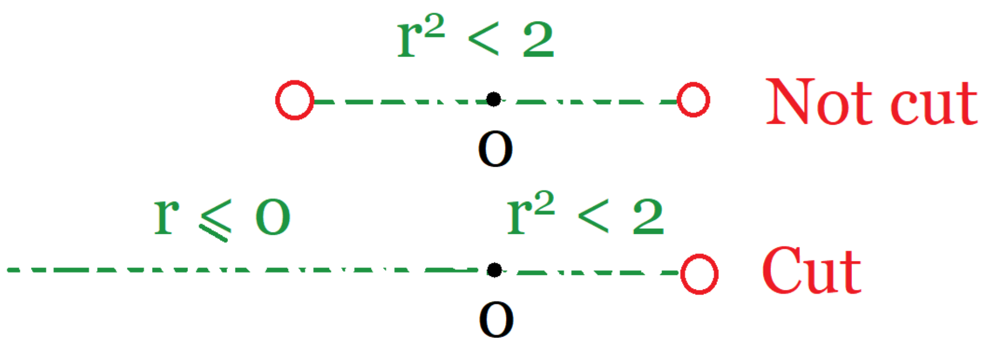
(c) $S = \left\{r \in \mathbb{Q} \mid r \leq \frac{1}{2}\right\}$: Doesn't satisfy (3) because S has a maximum of $\frac{1}{2}$: $r = \frac{1}{2} \in S$ but there is no $s \in S$ with $s > \frac{1}{2}$

Examples:

(a) $S = \left\{1, -2, \frac{4}{7}, 0, 1.2, 1.41, -3.6, \cdots\right\}$ (again some specific set of rational numbers, all that are $< \sqrt{2}$) is a cut, called $\sqrt{2}$ (but see a more concrete version below)

(b) $S = \left\{r \in \mathbb{Q} \,|\, r^3 < 2\right\}$ is a cut, called $\sqrt[3]{2}$.

(d) $S = \left\{r \in \mathbb{Q} \,|\, r^2 < 2\right\}$ is **NOT** a cut: $0 \in S$ and $-4 < 0$ but $-4 \notin S$. **BUT** $S = \{r \in \mathbb{Q} \,|\, r \leq 0\} \cup \left\{r \in \mathbb{Q} \,|\, r^2 < 2\right\}$ is a cut, called $\sqrt{2}$



Now that we defined what a cut is, let's see what operations we can do on them (just like operations on real numbers)

Definition:

If S and T are cuts, then

$$S + T = \{s + t \,|\, s \in S \text{ and } t \in T\}$$

Fact:

If S and T are cuts, then $S + T$ is a cut

Note: Multiplication of cuts is trickier to define; in particular $S \cdot T$ is **NOT** $\{st \,|\, s \in S \text{ and } t \in T\}$[5]

[5] See Pugh's book if you want to see how to define $S \times T$

Definition:

$\mathbb{R}$ = Set of all cuts in $\mathbb{Q}$

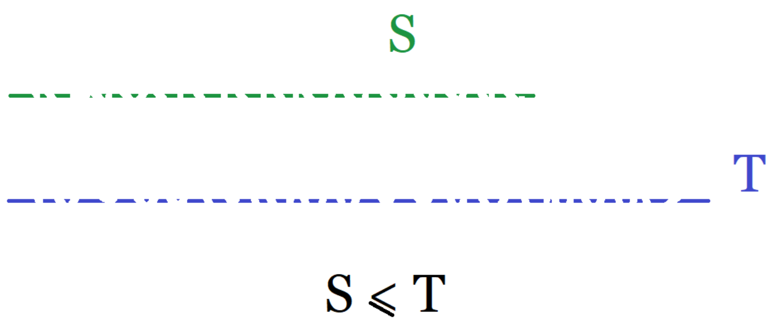
Fact:

With $+$ and $\cdot$ defined at above, $\mathbb{R}$ is a field (section 3)

Lastly, we can define an ordering on cuts simply as follows:

Definition:

If S and T are cuts, then $S \leq T$ means $S \subseteq T$



For example:

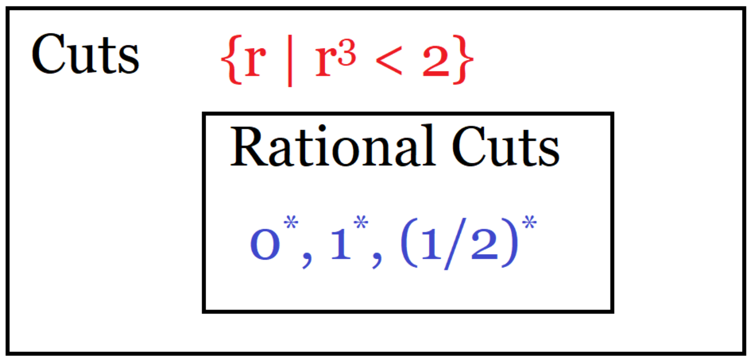
$$1^\star = \{r \in \mathbb{Q} \,|\, r < 1\} \subseteq \{r \in \mathbb{Q} \,|\, r < 2\} = 2^\star$$

So by definition $1^\star \leq 2^\star$

Fact:

With $\leq$ defined at above, $\mathbb{R}$ becomes an ordered field (section 3)

Since we can identify rational numbers a with rational cuts $a^\star$, with this identification, we can say that $\mathbb{R}$ "includes" $\mathbb{Q}$ (although, strictly speaking, elements of $\mathbb{R}$ are cuts, but elements of $\mathbb{Q}$ are rational numbers)



7. Optional: Proof of the Least Upper Bound Property

Video: Least Upper Bound Property Proof

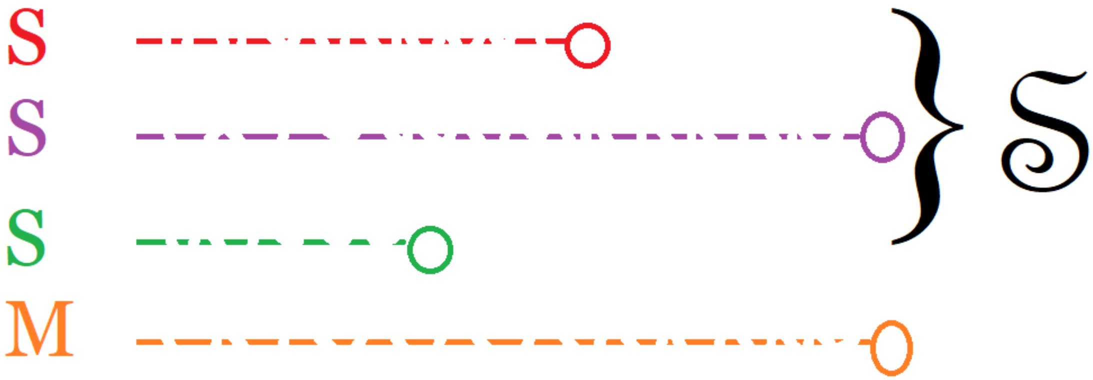
Why care about cuts? Because using them we can easily prove the least upper bound property. It's the elegance of the proof below that is the fruit of all our hard labor!

Least Upper Bound Property for Cuts:

If $\mathcal{S}$ is a nonempty set of cuts (= real numbers) that is bounded above, then $\mathcal{S}$ has a least upper bound.

Proof: Let M be the union of all the cuts $S \in \mathcal{S}$, that is:

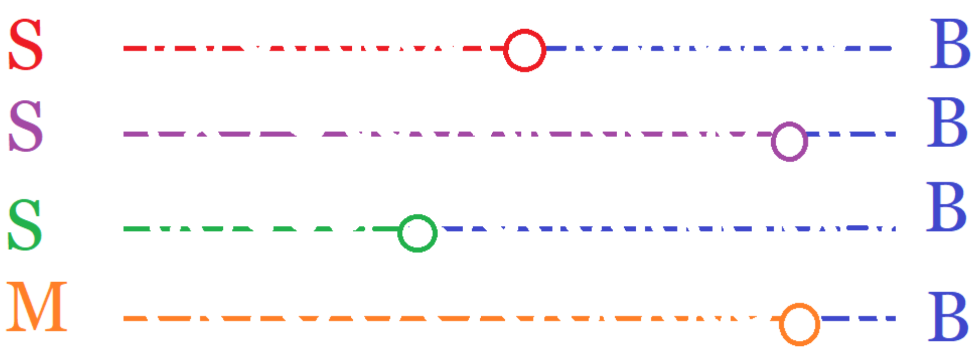
$$M = \{r \in \mathbb{Q} \,|\, r \in S \text{ for some } S \in \mathcal{S}\}$$



(STEP 1) **Claim:** M is a cut

First of all, $M \neq \emptyset$ because any cut S is nonempty (by definition) and M is just the union over all the cuts S.

Let's show $M \neq \mathbb{Q}$. Let B be an upper bound for $\mathcal{S}$ (which exists by our assumption). By definition of upper bound, for all $S \in \mathcal{S}$, $S \leq B$, meaning that $S \subseteq B$. Therefore, if you take the union $\cup S$ over all $S \in \mathcal{S}$, it is still true that $M = \cup S \subseteq B$, hence $M \subseteq B$ and since $B \neq \mathbb{Q}$ (Because $B < \infty$) we obtain $M \neq \mathbb{Q}$ ✓

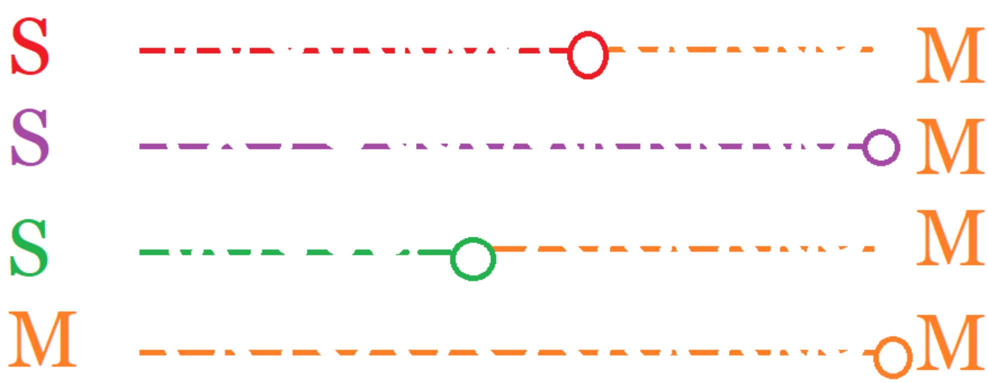


Suppose $r \in M$ and $s < r$. By definition of M, there is $S \in \mathcal{S}$ with $r \in S$. But since $s < r$ and S is a cut, we get $s \in S \subseteq M$, so $s \in M$. ✓

Suppose $r \in M$. By definition of M, there is $S \in \mathcal{S}$ with $r \in S$. But then since S is a cut, there is $s \in S$ with $s > r$. Since $S \subseteq M$, we get $s \in M$. So there is $s \in M$ such that $s > r$ ✓.

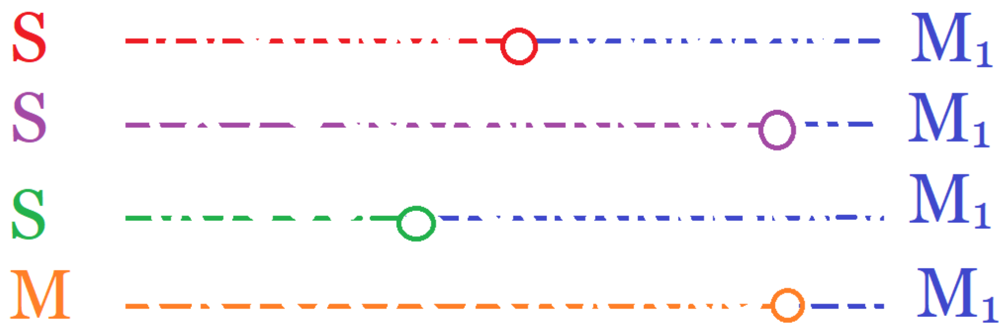
(STEP 2) **Claim:** M is an upper bound for $\mathcal{S}$.

This just follows from the definition of M as a union: Namely if $S \in \mathcal{S}$, then by definition of M, $S \subseteq M$. So for all $S \in \mathcal{S}$, $S \leq M$ ✓



(STEP 3) **Claim:** M is the least upper bound for $\mathcal{S}$

Let M_1 be any other upper bound for $\mathcal{S}$, meaning for all $S \in \mathcal{S}$, $S \leq M_1$, that is $S \subseteq M_1$. Then, if you take the union over all $S \in \mathcal{S}$, we get $\cup S \subseteq M_1$, that is $M \subseteq M_1$ (by definition of M), so $M \leq M_1$. This means that M is indeed the *least* upper bound: any other upper bound M_1 must be greater than or equal to M ✓ □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