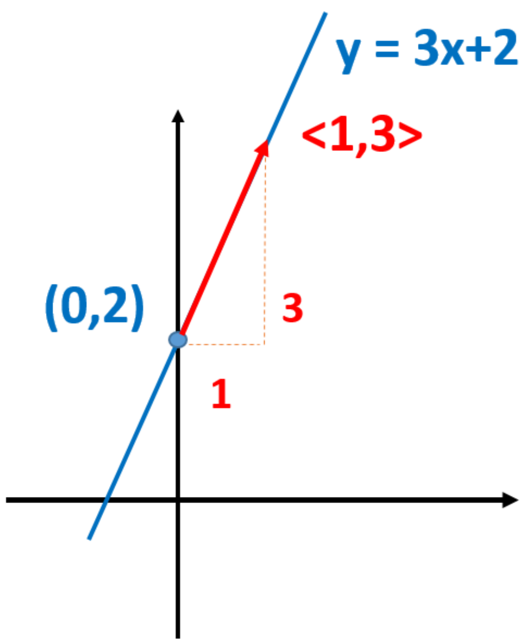
LECTURE 4: FUN WITH LINES

Today: We'll learn about a new way to describe lines, which is especially useful in this course.

1. Direction Vectors

Example 1: (Motivation)

Look at the line $y = 3x + 2$



Notice two things about the line:

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(1) It goes through a **point** $(0, 2)$

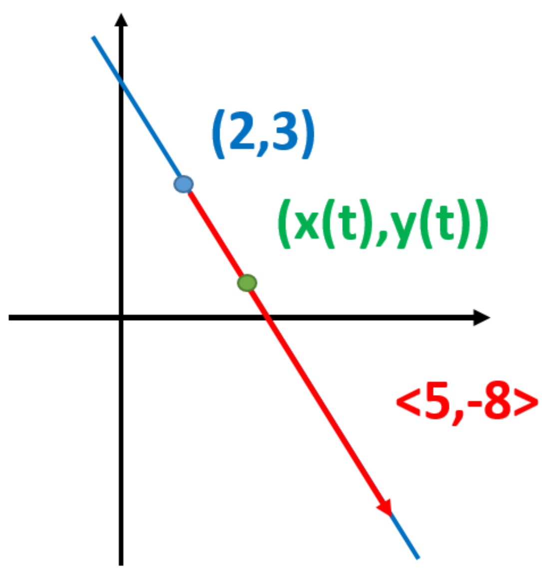
(2) It has **direction vector** $\langle 1, 3 \rangle$

(A direction vector is like a slope; tells you the direction a line is going).

Those two things characterize a line, as the next example shows:

Example 2

Find the equation of the line going through $(2, 3)$ and direction vector $\langle 5, -8 \rangle$



Main Idea: You're starting at the point $(2, 3)$ and moving forward and backward in the direction $\langle 5, -8 \rangle$, so we can naturally describe the line with **parametric equations**, as follows:

Fact:

The parametric equations of the line are given by:

$$\begin{cases} x(t) = 2 + 5t \\ y(t) = 3 - 8t \end{cases}$$

(Recall that parametric equations means that for every time t, you have a point $(x(t), y(t))$ on the line. Makes sense, since lines are one-dimensional)

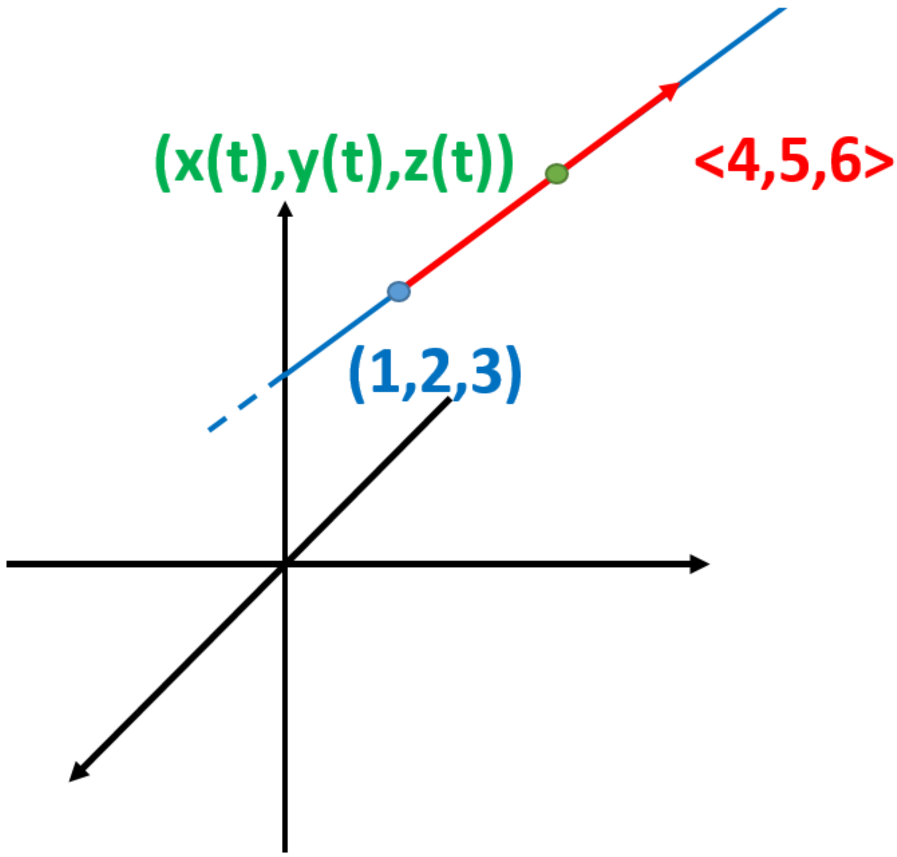
Note: This line can also be described in **vector form** as follows:

Vector Form:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle 2, 3 \rangle + t \langle 5, -8 \rangle = \langle 2 + 5t, 3 - 8t \rangle$$

Example 3: (Extra Practice)

Find parametric and vector equations of the line going through $(1.2, 3)$ and direction vector $\langle 4, 5, 6 \rangle$



Parametric Equations:

$$\begin{cases} x(t) = & 1 + 4t \\ y(t) = & 2 + 5t \\ z(t) = & 3 + 6t \end{cases}$$

Vector Equations:

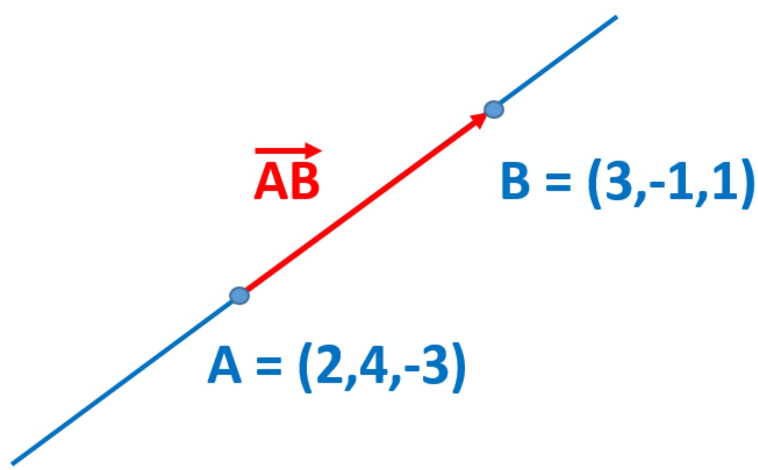
$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle 1, 2, 3 \rangle + t \langle 4, 5, 6 \rangle = \langle 1 + 4t, 2 + 5t, 3 + 6t \rangle$$

2. Some properties

Let's illustrate some properties via some examples.

Example 4: (Good quiz/exam question)

Find parametric equations of the line going through $A = (2, 4, -3)$ and $B = (3, -1, 1)$



(1) **Point:** $A = (2, 4, -3)$

(2) **Direction vector:**

$$\overrightarrow{AB} = \langle 3 - 2, -1 - 4, 1 + 3 \rangle = \langle 1, -5, 4 \rangle$$

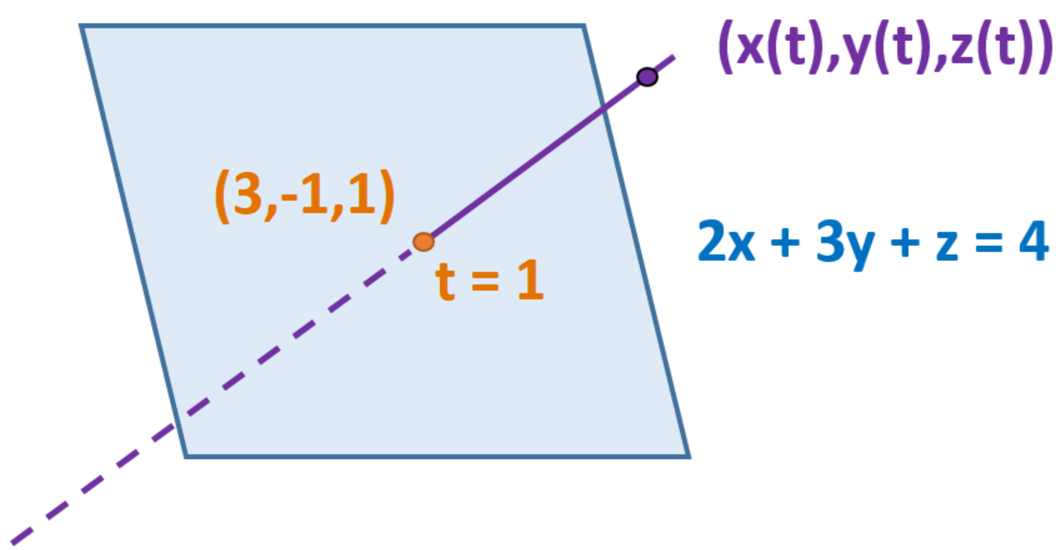
(3) **Answer:**

$$\begin{cases} x(t) = 2 + t \\ y(t) = 4 - 5t \\ z(t) = -3 + 4t \end{cases}$$

Example 5: (extra practice)

At which point does the line above intersect the following plane:

$$2x + 3y + z = 4$$



All you need to do is to plug in your formulas for x, y, z in the above equation:

$$\begin{aligned} 2x + 3y + z &= 4 \\ 2(2+t) + 3(4-5t) + (-3+4t) &= 4 \qquad \text{(see above)} \\ 4 + 2t + 12 - 15t - 3 + 4t &= 4 \\ -9t &= 4 - 4 - 12 + 3 \\ -9t &= -9 \\ t &= 1 \end{aligned}$$

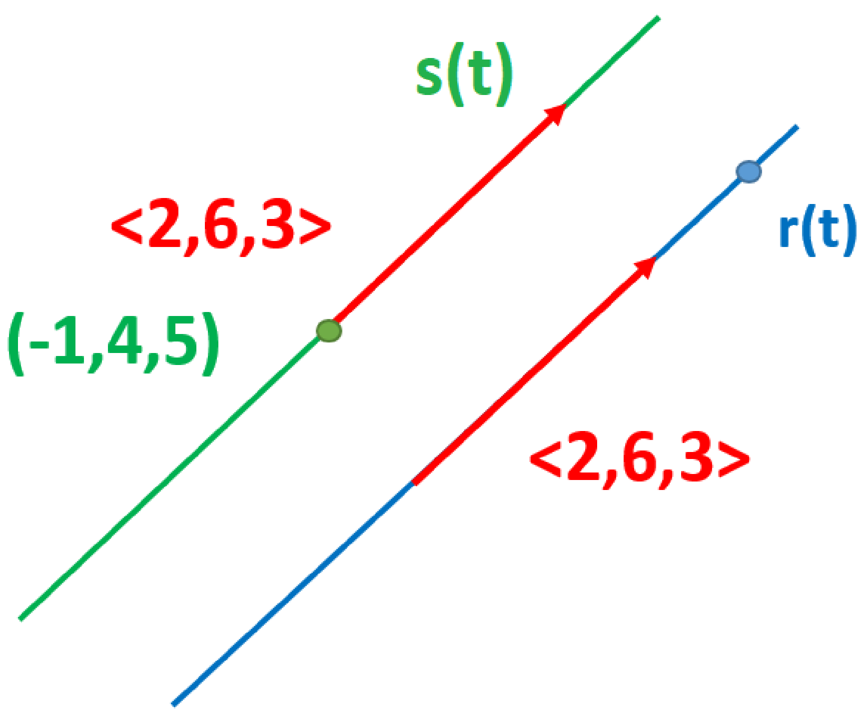
And so the point of intersection is:

$$\begin{cases} x(1) = 2 + 1 = 3 \\ y(1) = 4 - 5(1) = -1 \\ z(1) = -3 + 4(1) = 1 \end{cases}$$

Answer: $(3, -1, 1)$

Example 6:

Find the vector equations of the line going through $(-1, 4, 5)$ and parallel to the line with equation $\mathbf{r}(t) = \langle 2 + 2t, 6t, 1 + 3t \rangle$



(1) **Point:** $(-1, 4, 5)$

(2) **Direction vector:**

Since the two lines are parallel, they have the same direction vector, and so the direction vector is $\langle 2, 6, 3\rangle$

(3) Answer: $\mathbf{s}(t) = \langle -1 + 2t, 4 + 6t, 5 + 3t\rangle$

Example 7: (Symmetric Equations)

Find the **symmetric** equations of the line going through $(2, -3, -1)$ and direction vector $\langle 2, 2, -4\rangle$

STEP 1: First find the parametric equations:

$$\begin{cases} x(t) =& 2 + 2t \\ y(t) =& -3 + 2t \\ z(t) =& -1 - 4t \end{cases}$$

STEP 2: Solve for t in x, y, z:

$$\begin{cases} x = 2 + 2t \Rightarrow 2t = x - 2 \Rightarrow t = \dfrac{x-2}{2} \\ y = -3 + 2t \Rightarrow 2t = y + 3 \Rightarrow t = \dfrac{y+3}{2} \\ z = -1 - 4t \Rightarrow -4 = z + 1 \Rightarrow t = \dfrac{z+1}{-4} \end{cases}$$

STEP 3:

Fact: (Symmetric Equations)

The symmetric equations are

$$\frac{x-2}{2} = \frac{y+3}{2} = \frac{z+1}{-4}$$

More generally, if the point is (x_0, y_0, z_0) and the direction vector is $\langle a, b, c \rangle$, we get:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Note: Symmetric equations are the closest generalization to the equation $y = ax + b$ in two dimensions. But other than that they are kind of useless.

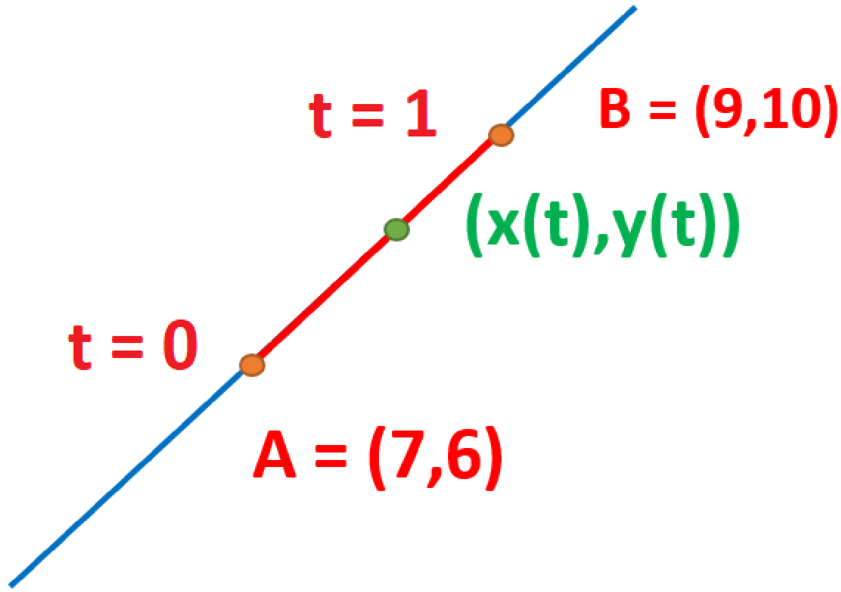
Oftentimes it will be useful to parametrize the **segment** between two points, instead of the whole line. For this, there is a nice shortcut.

Example 8: (Line Segment)

Find the parametric equations of the **line segment** between

$$A = (7, 6) \text{ and } B = (9, 10)$$

Method 1: Find the parametric equations



(1) **Point:** $(7, 6)$

(2) **Direction Vector:** $\overrightarrow{AB} = \langle 9 - 7, 10 - 6 \rangle = \langle 2, 4 \rangle$

(3) **Equations:**

$$\begin{cases} x(t) = 7 + 2t \\ y(t) = 6 + 4t \\ \qquad (0 \leq t \leq 1) \end{cases}$$

Note: The restriction $0 \leq t \leq 1$ is **important**, otherwise you're just describing the whole line. Indeed, you can check that at $t = 0$, you're at $(7, 6)$ and at $t = 1$, you're at $(9, 10)$.

Method 2: Faster Way (but basically the same as before)

$$\begin{cases} x(t) = (1-t)7 + t(9) \\ y(t) = (1-t)6 + t(10) \\ \qquad (0 \leq t \leq 1) \end{cases}$$

Note: Think of $(1-t)$ and t as on/off switches:

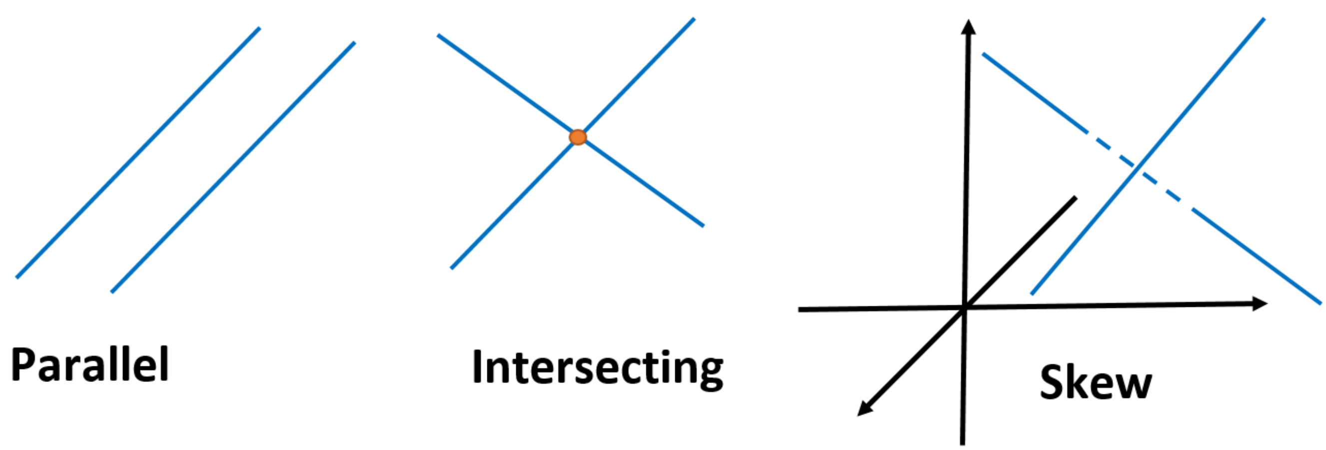
At $t = 0$, we have $(1-t) = 1$ (on) and $t = 0$ (off), so we're at $(7, 6)$

At $t = 1$, we have $(1-t) = 0$ (off) and $t = 1$ (on), so we're at $(9, 10)$.

3. Skew Lines

Finally, here is a nice application of direction vectors.

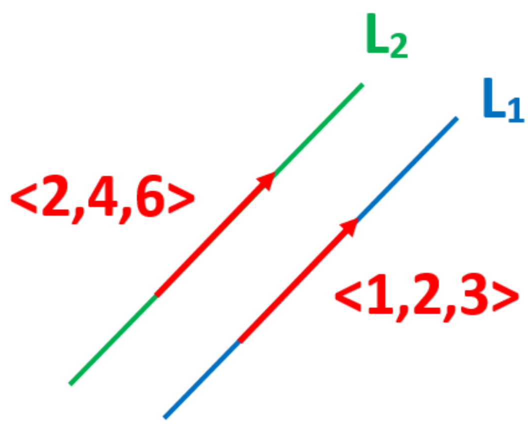
In 3 dimensions, lines can be either **parallel**, **intersecting**, or **skew** (neither parallel, nor intersecting):



Example 9: (Good quiz/exam question)

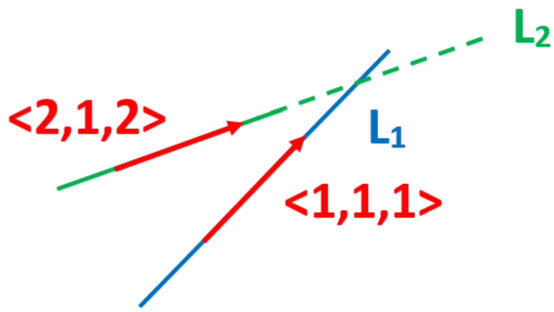
Are the following lines parallel, intersecting, or skew?

$$(a) \quad L_1 : \begin{cases} x(t) = t+1 \\ y(t) = 2t+3 \\ z(t) = 3t-4 \end{cases} \qquad L_2 : \begin{cases} x(s) = 2s+3 \\ y(s) = 4s-5 \\ z(s) = 6s-8 \end{cases}$$



L_1 has direction vector $\langle 1, 2, 3\rangle$ and L_2 has direction vector $\langle 2, 4, 6\rangle$. Since the direction vectors are parallel, hence the lines are **parallel**

$$(b) \quad L_1 : \begin{cases} x(t) = t+1 \\ y(t) = t+2 \\ z(t) = t+3 \end{cases} \qquad L_2 : \begin{cases} x(s) = 2s \\ y(s) = s+1 \\ z(s) = 2s+1 \end{cases}$$



The direction vector of L_1 is $\langle 1,1,1\rangle$ and the direction vector of L_2 is $\langle 2,1,2\rangle$ which are not parallel, so the lines either intersect or are skew. To figure this out, we need to solve all 3 equations.

$$\begin{cases} t+1 = 2s \\ t+2 = s+1 \\ t+3 = 2s+1 \end{cases}$$

Here it's easiest to solve each equation one at a time. The first equation tells us $t = 2s - 1$

Hence the second equation becomes:

$$\begin{aligned} t+2 &= s+1 \\ 2s-1+2 &= s+1 \\ 2s+1 &= s+1 \\ s &= 0 \end{aligned}$$

And therefore $t = 2s - 1 = 2(0) - 1 = -1$

But then the third equation becomes:

$$t + 3 = 2s + 1 \Rightarrow (-1) + 3 = 2(0) + 1 \Rightarrow 2 = 1$$

Which is a contradiction, and therefore the equation has no solution, and hence the lines are **skew** (= not parallel and they don't intersect).

$$(c) \quad L_1 : \begin{cases} x(t) = t+1 \\ y(t) = t-2 \\ z(t) = t-3 \end{cases} \qquad L_2 : \begin{cases} x(s) = 2s-1 \\ y(s) = 3s-5 \\ z(s) = 4s-7 \end{cases}$$

L_1 has direction vector $\langle 1, 1, 1 \rangle$ and L_2 has direction vector $\langle 2, 3, 4 \rangle$, which are not parallel.

Here again, let's determine the intersection (if any) by solving the equations:

$$\begin{cases} t+1 = 2s-1 \\ t-2 = 3s-5 \\ t-3 = 4s-7 \end{cases}$$

From the first equation, we get $t = (2s-1) - 1 = 2s - 2$

Then the second equation becomes:

$$\begin{aligned} t-2 &= 3s-5 \\ 2s-2-2 &= 3s-5 \\ 2s-4 &= 3s-5 \\ s &= -4+5 \\ s &= 1 \end{aligned}$$

And so $t = 2(1) - 2 = 0$, hence $\boxed{\text{s = 1, t = 0}}$

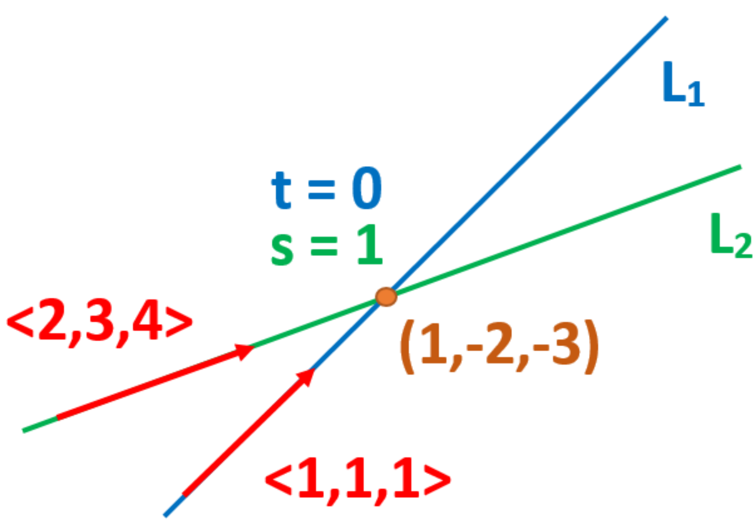
And the third equation becomes:

$$\begin{aligned} t - 3 &= 4s - 7 \\ 0 - 3 &= 4(1) - 7 \\ -3 &= -3 \checkmark \end{aligned}$$

This means the points intersect, and to find the point of intersection, you can just let $t = 0$ in L_1:

$$\begin{cases} x(0) = 0 + 1 = 1 \\ y(0) = 0 - 2 = -2 \\ z(0) = 0 - 3 = -3 \end{cases}$$

Answer: L_1 and L_2 intersect at the point $\boxed{(1, -2, -3)}$



Check: (Optional) To check your work, you can let $s = 1$ in L_2 and see if it gives you the same point:

$$\begin{cases} x(1) =& 2(1) - 1 = 1 \\ y(1) =& 3(1) - 5 = -2 \\ z(1) =& 4(1) - 7 = -3 \end{cases}$$

Summary: The following flowchart summarizes all the scenarios and how to check them: