LECTURE 4: FIXED POINTS AND ODE

1. The Banach Fixed Point Theorem

Video: Banach Fixed Point Theorem

Definition: p is a **fixed point** of f if $f(p) = p$

That is, p does not change when you apply f to it.

Question: When does a function have a fixed point?

Let (X, d) be a metric space

Definition: $f : X \to X$ is a **contraction** if there is $k < 1$ such that for all x, y, we have

$$d(f(x), f(y)) \leq kd(x, y)$$

Intuitively, f shrinks distances between points. Notice contractions are continuous.

Theorem: [Banach Fixed Point Theorem] If X is complete and f is a contraction, then f has a unique fixed point p.

Analogy: You may have noticed this phenomenon when you start with a number on a calculator, and repeatedly apply cos or $\sqrt{x}$ on it.

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Eventually the number stays the same!

Proof:[1]

STEP 1: Let $x_0 \in X$ and define $x_n = f^n(x_0)$ (f applied n times)

Notice $d(x_1, x_2) = d(f(x_0), f(x_1)) \leq kd(x_0, x_1)$ and

$d(x_2, x_3) = d(f(x_1), f(x_2)) \leq kd(x_1, x_2) \leq kkd(x_0, x_1) = k^2 d(x_0, x_1)$

And more generally you can show that

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$$

STEP 2: **Claim:** (x_n) is Cauchy

Why? Let $\epsilon > 0$ be given and N be TBA, then if $m, n > N$ (WLOG assume $n \geq m$), then

$$\begin{aligned} d(x_m, x_n) \leq & d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ \leq & k^m d(x_0, x_1) + k^{m+1} d(x_0, x_1) + \cdots + k^{n-1} d(x_1, x_0) \qquad \text{(By STEP 1)} \\ \leq & \left(k^m + k^{m+1} + \cdots + k^{n-1}\right) d(x_1, x_0) \\ = & k^m \left(1 + k + \cdots + k^{n-m-1}\right) d(x_0, x_1) \\ \leq & k^m \left(1 + k + k^2 + \cdots\right) d(x_0, x_1) \\ = & k^m \left(\frac{1}{1-k}\right) d(x_0, x_1) \\ \leq & \frac{k^N}{1-k} d(x_0, x_1) \qquad \text{Since } m > N \text{ and } k < 1 \end{aligned}$$

[1] The proof is from Pugh's book, Theorem 23 in Chapter 4

But since $k < 1$ we have $\lim_{n\to\infty} k^n = 0$, so we can choose N large enough so that $\frac{k^N}{1-k} d(x_0, x_1) < \epsilon$, which in turn implies $d(x_m, x_n) < \epsilon$ ✓

STEP 3: Since (x_n) is Cauchy and X is complete, (x_n) converges to some p

Claim: p is a fixed point of f.

This follows because

$$\begin{aligned} x_{n+1} &= f(x_n) \\ \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} f(x_n) \\ p &= f\left(\lim_{n\to\infty} x_n\right) \qquad \text{(continuity)} \\ p &= f(p) \checkmark \end{aligned}$$

STEP 4: Uniqueness: Suppose there are two fixed points $p \neq q$, then

$$d(p, q) = d(f(p), f(q)) \leq k d(p, q) < d(p, q)$$

Which is a contradiction □

2. Application to ODE

As an application, we can prove the celebrated ODE existence-uniqueness theorem. Consider the ODE:

$$\begin{cases} y' = f(y) \\ y(0) = y_0 \end{cases}$$

Recall: f is **Lipschitz** if there is $L > 0$ such that for every x, y, we have $|f(x) - f(y)| \leq L |x - y|$

Theorem: [Picard-Lindelöf] If f is Lipschitz and $y_0 \in \mathbb{R}$, then for some small $\tau > 0$, there exists a solution $y : [-\tau, \tau] \to \mathbb{R}$ of the ODE

Note: The solution is "locally unique," in the sense below.

Proof:[2]

STEP 1: Main Observation: By integrating the ODE, it is equivalent to

$$\begin{aligned} \int_0^t y'(s)ds &= \int_0^t f(y(s))ds \\ y(t) - y_0 &= \int_0^t f(y(s))ds \\ y(t) &= y_0 + \int_0^t f(y(s))ds \end{aligned}$$

STEP 2: Let $\tau > 0$ TBA

Since f is continuous, it is bounded around y_0: There is some $r > 0$ and $C > 0$ such that $|f(x)| \leq C$ for all $x \in [y_0 - r, y_0 + r]$.

Let X be the space of continuous functions $y : [-\tau, \tau] \to [y_0 - r, y_0 + r]$ with the sup norm.

Given $y \in X$, define $\Phi(y) \in X$ (to be shown) by

$$\Phi(y)(t) = y_0 + \int_0^t f(y(s))ds$$

We're done once we show that Φ has a fixed point y, because then $\Phi(y) = y$ and we get

[2] The proof is a simplified version of the one in Theorem 24 of Pugh's book

$$y(t) = y_0 + \int_0^t f(y(s))ds \checkmark$$

STEP 3: Proof that Φ is a contraction

First show that $\Phi : X \to X$: Notice that if y is continuous, then $\int_0^t f(y)$ is continuous (in fact differentiable) and hence $\Phi(y)(t)$ is continuous. Moreover

$$|\Phi(y)(t) - y_0| = \left| \int_0^t f(y(s))ds \right| \leq \int_0^t |f(y)|\, ds \leq \int_0^t C ds = Ct \leq C\tau \leq r$$

Provided you choose τ such that $\tau C \leq r$

Hence $\Phi(y) \in [y_0 - r, y_0 + r]$ and so $\Phi(y) \in X$.

Moreover, Φ is a contraction because

$$\begin{aligned}
d(\Phi(y), \Phi(z)) =& \sup_t \left| y_0 + \int_0^t f(y(s))ds - \left(y_0 + \int_0^t f(z(s))ds \right) \right| \\
\leq& \sup_t \left| \int_0^t f(y(s)) - f(z(s))ds \right| \\
\leq& \sup_t \int_0^t |f(y(s)) - f(z(s))|\, ds \\
\leq& \int_0^\tau |f(y(s)) - f(z(s))|\, ds \text{ (the integral is increasing in } t) \\
\leq& \int_0^\tau \left(\sup_s |f(y(s)) - f(z(s))| \right) ds \\
=& \left(\sup_s |f(y(s)) - f(z(s))| \right) \int_0^\tau 1 \\
\leq& L \sup_s |y(s) - z(s)|\, \tau \\
=& L\tau d(y, z)
\end{aligned}$$

This becomes a contraction provided we choose τ so that $L\tau < 1$

STEP 4: Uniqueness

Any other solution $z(t)$ is also a fixed point of Φ, that is $\Phi(z) = z$. Since a contraction has a unique fixed point, we have $z = y$. This is what local uniqueness means. □

3. Nowhere Differentiable Function

As a nice application of the ideas learned in this chapter, let's construct a function that is continuous but *nowhere* differentiable.

Theorem: There exists a continuous function on $\mathbb{R}$ that is differentiable nowhere.

STEP 1: Start with $\phi(x) = |x|$ on $[-1, 1]$ and extend it periodically on $\mathbb{R}$ such that $\phi(x + 2) = \phi(x)$ (see picture in lecture)

Then for all x and y, we have

$$|\phi(x) - \phi(y)| \leq |x - y|$$

Why? If $x, y \in [-1, 1]$ this follows from $||x| - |y|| \leq |x - y|$, and for general x, y, we can reduce to this case by periodicity.

In particular, ϕ is continuous on $\mathbb{R}$

STEP 2: Define f as:

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k \phi\left(4^k x\right)$$

f is the superposition of zig-zag functions that become smaller but crazier (see picture in lecture)

Claim: The series defining f converges uniformly

Why? Beautiful application of the Weierstraß M-test: Since $|\phi| \leq 1$

$$\left|\left(\frac{3}{4}\right)^k \phi\left(4^k x\right)\right| = \left|\frac{3}{4}\right|^k \underbrace{\left|\phi\left(4^k x\right)\right|}_{\leq 1} \leq \underbrace{\left(\frac{3}{4}\right)^k}_{M_k}$$

And $\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k$ converges because it's a geometric series.

In particular, since f is the uniform limit of the continuous partial sums, f is continuous.

STEP 3: **Claim:** f is nowhere differentiable.

Idea: If f were differentiable at x, then for every sequence $s_n \to 0$, $\frac{f(x+s_n)-f(x)}{s_n}$ would converge (to $f'(x)$). But we will choose a clever s_n that will make this diverge.

Fix x and n and let

$$s_n = \pm\left(\frac{1}{2}\right)4^{-n}$$

The sign is TBA. Notice $s_n \to 0$.

Consider the difference quotient

$$\frac{f(x+s_n)-f(x)}{s_n} = \sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k \underbrace{\left(\frac{\phi\left(4^k\left(x+s_n\right)\right)-\phi\left(4^k x\right)}{s_n}\right)}_{a_k}$$

(we can rearrange the sum this way because of uniform convergence)

There are three cases now depending on whether k is small or large.

Case 1: $k > n$ (k is large)

$$4^k\left(x+s_n\right)-4^k x = \cancel{4^k x}+4^k s_n-\cancel{4^k x} = 4^k\left(\pm\left(\frac{1}{2}\right)4^{-n}\right) = \pm\underbrace{\left(\frac{1}{2}\right)4^{k-n}}_{\text{Even integer}}$$

By periodicity of ϕ, we get $\phi(4^k(x+s_n))-\phi(4^k x)=0$, and so $a_k=0$

Case 2: $k < n$ (k small) Then using the Lipschitz condition on ϕ:

$$|a_k| = \frac{\left|\phi\left(4^k\left(x+s_n\right)\right)-\phi\left(4^k x\right)\right|}{|s_n|} \leq \frac{\left|4^k(x+s_n)-4^k x\right|}{|s_n|} = \frac{4^k\left|s_n\right|}{|s_n|} = 4^k$$

Case 3: $k = n$ (k medium)

In that case $4^k\left(x+s_n\right)=4^k x\pm\frac{1}{2}$.

Now choose the sign $\pm$ in such a way that there are no integers between $4^k x\pm\frac{1}{2}$ and $4^k x$. In that case ϕ is linear on that interval and we get

$$|a_n| = \frac{\left|\phi\left(4^k x\pm\frac{1}{2}\right)-\phi\left(4^k x\right)\right|}{\left|\pm\frac{1}{2}\left(4^{-n}\right)\right|} = \frac{\left|4^k x\pm\frac{1}{2}-4^k x\right|}{\frac{1}{2}\left(4^{-n}\right)} = \frac{\frac{1}{2}}{\frac{1}{2}\left(4^{-n}\right)} = 4^n$$

STEP 4: Therefore by the 3 cases we have

$$\begin{aligned}
\left|\frac{f(x+s_n)-f(x)}{s_n}\right| &= \left|\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k a_k\right| \\
&= \left|\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k a_k + \left(\frac{3}{4}\right)^n a_n + \sum_{k=n+1}^{\infty}\left(\frac{3}{4}\right)^k 0\right| \\
&= \left|\left(\frac{3}{4}\right)^n a_n - \left(-\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k a_k\right)\right| \\
&\geq \left|\left|\left(\frac{3}{4}\right)^n a_n\right| - \left|-\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k a_k\right|\right| \quad \text{Reverse triangle inequality}
\end{aligned}$$

$$\text{But } \left|\left(\frac{3}{4}\right)^n a_n\right| = \left(\frac{3}{4}\right)\underbrace{|a_n|}_{4^n} = 3^n$$

$$\text{And } \left|-\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k a_k\right| \leq \sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k \underbrace{|a_k|}_{\leq 4^k} \leq \sum_{k=0}^{n-1} 3^k = \frac{3^n-1}{3-1} = \frac{1}{2}\left(3^n-1\right)$$

Therefore

$$\left|\frac{f(x+s_n)-f(x)}{s_n}\right| \geq \left|3^n - \frac{1}{2}\left(3^n\right) + \frac{1}{2}\right| = \frac{3^n+1}{2} \stackrel{n\to\infty}{\to} \infty$$

So the limit does not exist, and f is not differentiable at x □

Aside: This function f is not the exception, but the rule! In general, the “generic” function is nowhere differentiable: If you pick a continuous function at random, chances are that it's nowhere differentiable.