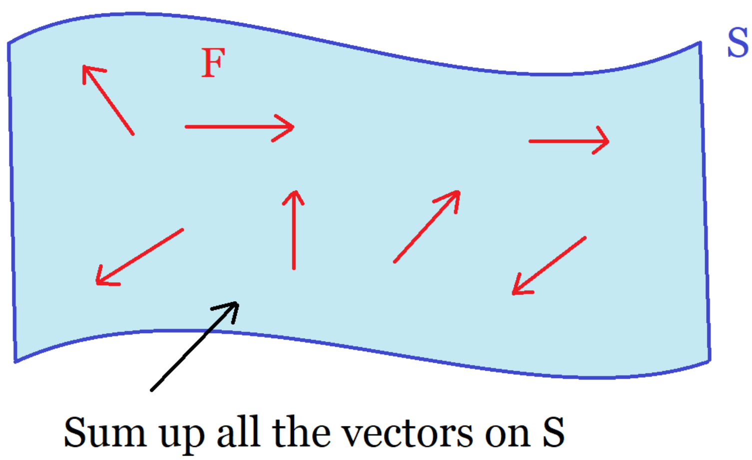
LECTURE 40: SURFACE INTEGRALS (II)

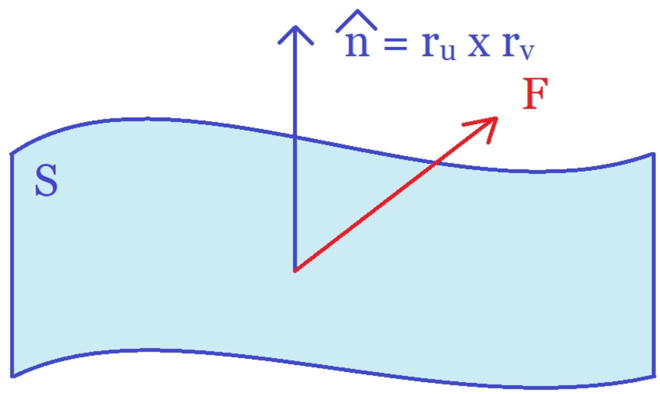
1. Surface Integrals of Vector Fields

Goal: Given a vector field F and a surface S, want to sum up the values of F over S



For line integrals, we dotted F with the **tangent** vector $\mathbf{r}'(t)$, this time we dot F with the **normal** vector $\hat{n}$

Date: Friday, December 3, 2021.

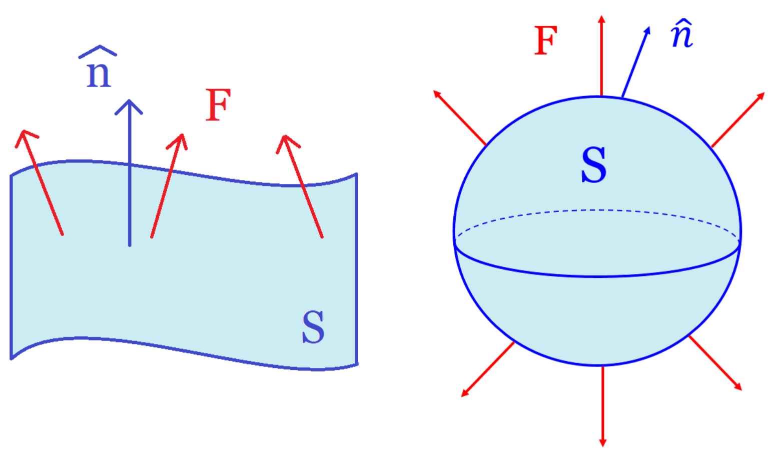


Surface Integral of F

$$\int\int_S F \cdot d\mathbf{S} = \int\int F \cdot \hat{n} = \int\int_D F(r(u,v)) \cdot \underbrace{(r_u \times r_v)}_{\hat{n}} dudv$$

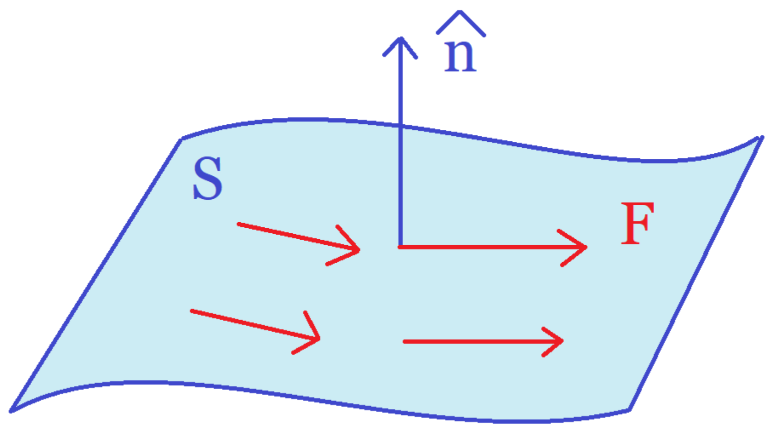
Application: In Physics, $\int\int_S F \cdot d\mathbf{S}$ is called the **net flux** of F across S, measures how much F flows in or out of S:

Scenario 1:



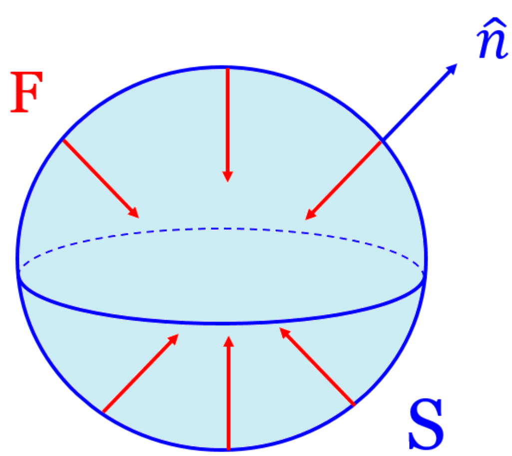
Here $\int\int_S F \cdot d\mathbf{S} > 0$, F flows **out** of S, think water leaking out

Scenario 2:



Here $F \cdot \hat{n} = 0$ so $\int\int_S F \cdot d\mathbf{S} = 0$, F is tangent to S

Example 3: $\int\int_S F \cdot d\mathbf{S} < 0$, F flows **into** S



2. Example

Video: Surface Integral of a Vector Field

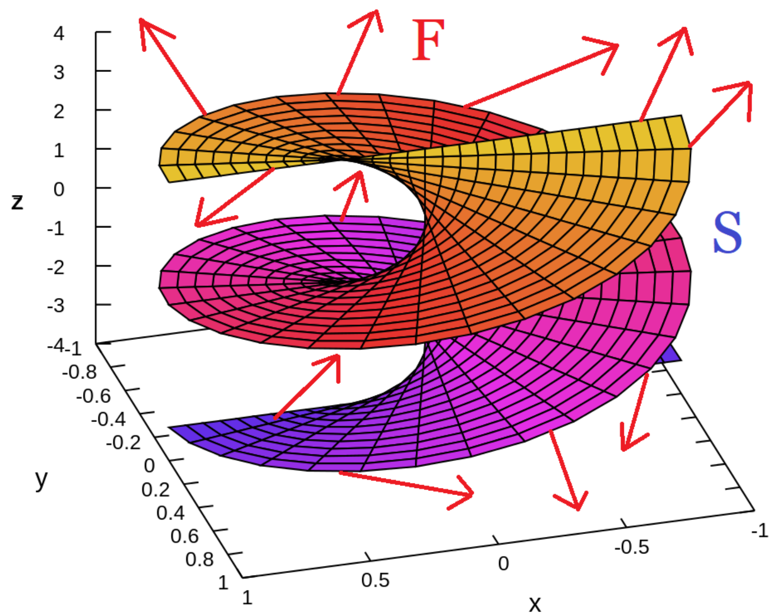
Example 1:

$$\int\int_S F \cdot d\mathbf{S}$$

$F = \left\langle 2x, 2y, z^2 \right\rangle$ and S is the helicoid parametrized by

$$r(u, v) = \langle u\cos(v), u\sin(v), v\rangle$$
$$0 \leq u \leq 1$$
$$0 \leq v \leq \pi$$

STEP 1: **Picture:**



STEP 2: Slopes

$$\begin{aligned} r_u &= \langle \cos(v), \sin(v), 0 \rangle \\ r_v &= \langle -u\sin(v), u\cos(v), 1 \rangle \end{aligned}$$

STEP 3: Normal Vector

$$\begin{aligned} \hat{n} = r_u \times r_v &= \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix} \\ &= \left\langle \sin(v), -\cos(v), u\cos^2(v) + u\sin^2(v) \right\rangle \\ &= \langle \sin(v), -\cos(v), u \rangle \end{aligned}$$

STEP 4: Integrate

$$\begin{aligned} &\int\int_S F \cdot d\mathbf{S} \\ =&\int\int_D F \cdot (r_u \times r_v)\, dudv \\ =&\int\int_D \underbrace{\left\langle 2u\cos(v), 2u\sin(v), v^2 \right\rangle}_{\langle 2x, 2y, z^2 \rangle} \cdot \underbrace{\langle \sin(v), -\cos(v), u \rangle}_{r_u \times r_v} dudv \\ =&\int_0^\pi \int_0^1 \cancel{2u\cos(v)\sin(v)} - \cancel{2u\sin(v)\cos(v)} + v^2 ududv \\ =&\int_0^\pi \int_0^1 v^2 ududv \\ =&\left(\int_0^1 udu \right) \left(\int_0^\pi v^2 dv \right) \\ =&\left(\frac{1}{2} \right) \left(\frac{\pi^3}{3} \right) \\ =&\frac{\pi^3}{6} \end{aligned}$$

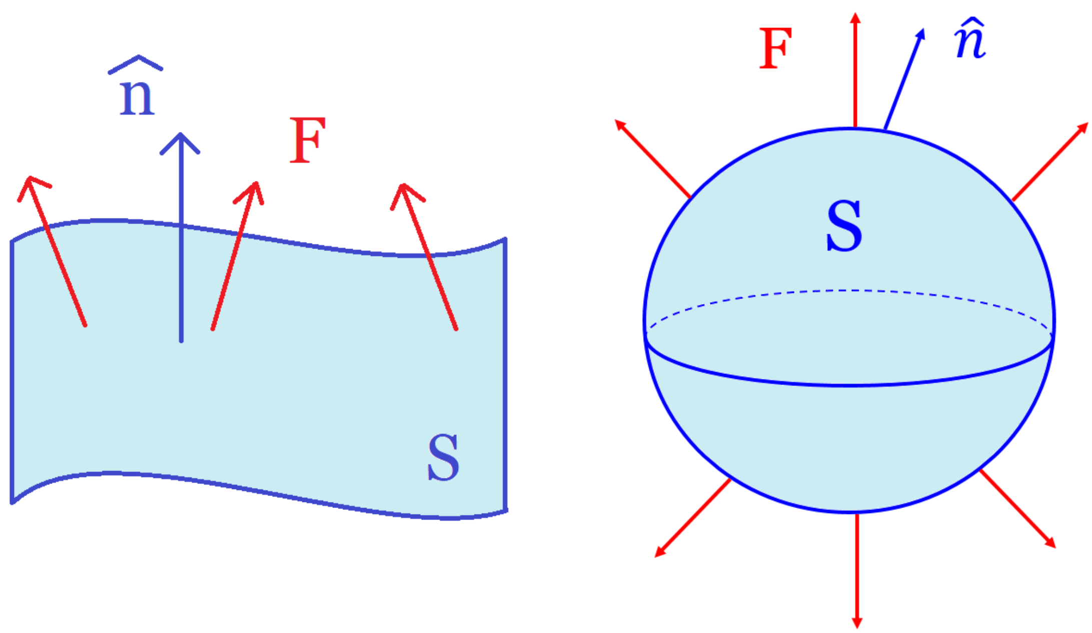
3. Orientation

Warning

Orientation matters! Make sure $\hat{n} = r_u \times r_v$ points:

(1) **Upwards** (for graphs)

(2) **Outwards** (for closed surfaces like spheres)

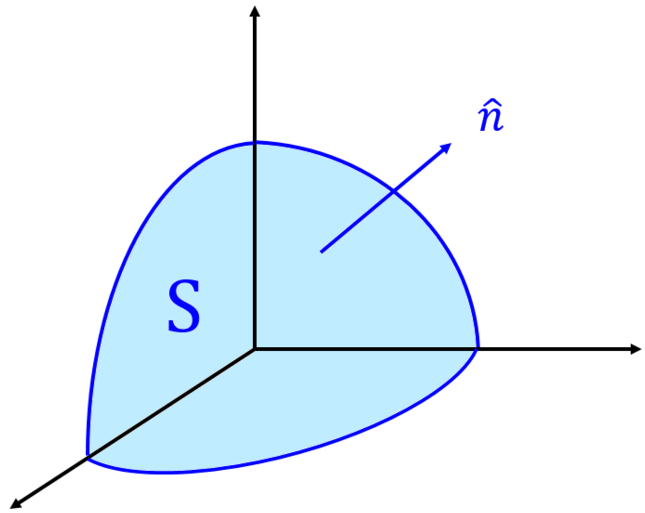


Rule of Thumb: Usually, but not always, check that the z component is ≥ 0 (but best to use a picture)

Example 2:

Calculate the net flux of $F = \langle 0, 0, z \rangle$ across the surface S, where S is the Sphere of radius 1 in the first octant

STEP 1: Picture:



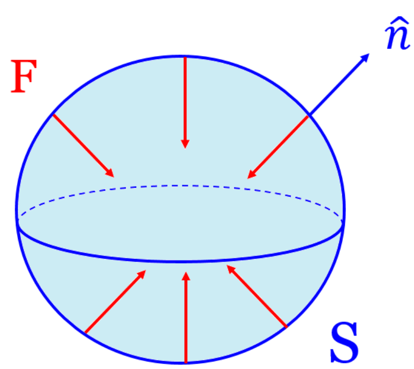
STEP 2: Parametrize:

$$r(\theta, \phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle \qquad 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \phi \leq \frac{\pi}{2}$$

STEP 3: Normal Vector: From the previous lecture, found

$$\hat{n} = r_\theta \times r_\phi = \left\langle -\sin^2(\phi)\cos(\theta), -\sin^2(\phi)\sin(\theta), \underbrace{-\sin(\phi)\cos(\phi)}_{\leq 0} \right\rangle$$

⚠ $\hat{n}$ points inwards, not outwards!



Solution: Use $-\hat{n}$ instead:

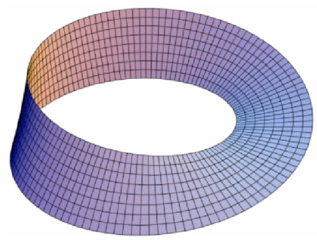
$$\hat{n} = \left\langle +\sin^2(\phi)\cos(\theta), +\sin^2(\phi)\sin(\theta), \underbrace{+\sin(\phi)\cos(\phi)}_{\geq 0} \right\rangle$$

STEP 4: Integrate

$$\begin{aligned}
&\int\int_S F\cdot d\mathbf{S}\\
=&\int\int_D F\cdot\hat{n}\,d\theta d\phi\\
=&\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\underbrace{\langle 0,0,\cos(\phi)\rangle}_{\langle 0,0,z\rangle}\cdot\underbrace{\left\langle\sin^2(\phi)\cos(\theta),\sin^2(\phi)\sin(\theta),\sin(\phi)\cos(\phi)\right\rangle}_{\hat{n}}\,d\theta d\phi\\
=&\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin(\phi)\cos^2(\phi)d\theta d\phi\\
=&\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\cos^2(\phi)\sin(\phi)d\phi \qquad (u=\cos(\phi))\\
=&\frac{\pi}{2}\left[-\frac{1}{3}\cos^3(\phi)\right]_0^{\frac{\pi}{2}}\\
=&\left(\frac{\pi}{2}\right)\left(\frac{1}{3}\right)\\
=&\frac{\pi}{6}
\end{aligned}$$

Interesting Fact: There is a surface called the Möbius strip, where $\hat{n}$ changes orientation, meaning that it goes from outside to inside! This surface has no sides and is called a non-orientable surface. Needless to say, but you cannot evaluate $\int\int_S F\cdot d\mathbf{S}$ on it! [1]

[1] Picture courtesy Science News



Why did the chicken cross the Möbius strip? To get to the same side!

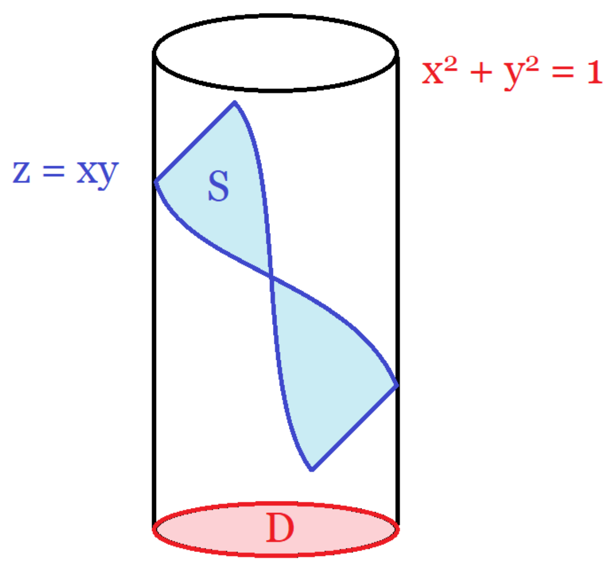
4. The case of Functions

Example 3:

$$\int\int_S F \cdot d\mathbf{S} \qquad F = \langle y, x, 3z \rangle$$

S : Graph of $z = xy$ over the disk of radius 1

STEP 1: Picture:



STEP 2: Parametrize

$$r(x, y) = \langle x, y, xy \rangle$$

STEP 3: Slopes

$$r_x = \langle 1, 0, (xy)_x \rangle = \langle 1, 0, y \rangle$$
$$r_y = \left\langle 0, 1, (xy)_y \right\rangle = \langle 0, 1, x \rangle$$

STEP 4: Normal Vector

$$\hat{n} = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \left\langle -y, -x, \underbrace{1}_{\geq 0} \right\rangle \checkmark \quad \text{Upwards (for graphs)}$$

STEP 5: Integrate

$$\begin{aligned}
\int\int_S F \cdot d\mathbf{S} &= \int\int_D F \cdot (r_x \times r_y)\, dxdy \\
&= \int\int_D \underbrace{\langle y, x, 3xy \rangle}_{\langle y,x,3z \rangle} \cdot \underbrace{\langle -y, -x, 1 \rangle}_{\hat{n}} dxdy \\
&= \int\int_D -y^2 - x^2 + 3xydxdy \\
&= \int_0^{2\pi} \int_0^1 \left(-r^2 + 3r\cos(\theta) r \sin(\theta)\right) rdrd\theta \\
&= \int_0^{2\pi} \int_0^1 -r^3 + 3r^3 \cos(\theta)\sin(\theta) drd\theta
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \int_0^1 r^3 \left(-1 + 3\cos(\theta)\sin(\theta)\right) dr d\theta \\
&= \left(\int_0^1 r^3 dr\right) \left(\int_0^{2\pi} -1 + 3\cos(\theta)\sin(\theta) d\theta\right) \\
&= \left[\frac{r^4}{4}\right]_0^1 \left[-\theta + 3\left(\frac{\sin^2(\theta)}{2}\right)\right]_0^{2\pi} \quad (u = \sin(\theta)) \\
&= \frac{1}{4}(-2\pi) \\
&= -\frac{\pi}{2}
\end{aligned}
$$

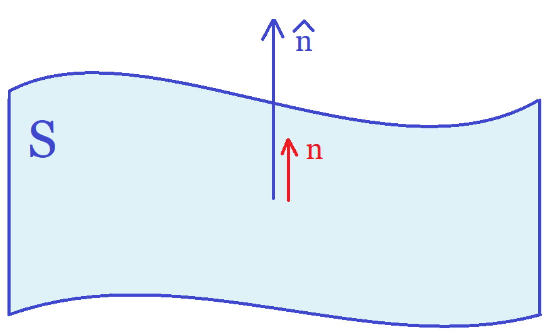
There is again an explicit formula for $\int\int_S F \cdot d\mathbf{S}$ when S is the graph of a function, but please don't memorize it; just do it the way above.

5. Two Surface Integrals

Question: Are $\int\int_S F \cdot d\mathbf{S}$ and $\int\int_S f dS$ related? Yes!

Definition

$$n = \frac{\hat{n}}{\|\hat{n}\|} = \textbf{ Unit } \text{normal vector (Length} = 1)$$

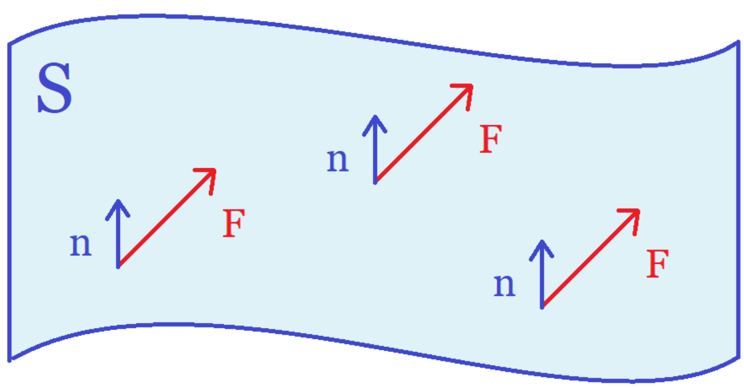


Let's look again at our surface integral:

$$\begin{aligned}
\int\int_S F \cdot d\mathbf{S} &\overset{\text{DEF}}{=} \int\int_D F \cdot \hat{n}\, dudv \\
&\overset{\text{TRICK}}{=} \int\int_D F \cdot \underbrace{\frac{\hat{n}}{\|\hat{n}\|}}_{n} \|\hat{n}\| dudv \\
&\overset{DEF}{=} \int\int_D F \cdot n \underbrace{\|r_u \times r_v\|\, dudv}_{dS} \\
&\overset{DEF}{=} \int\int_S F \cdot n \;\, dS
\end{aligned}$$

Adult Surface Integral

$$\int\int_S F \cdot d\mathbf{S} = \int\int_S F \cdot n\, dS$$



So the surface integral of the **vector field** F is the surface integral of the **function** $F \cdot n$. This again expresses the fact that we're summing up the values of F over the surface S.