LECTURE 41: DIVERGENCE THEOREM

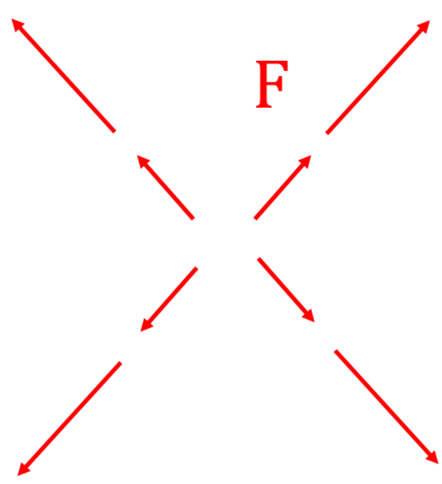
Welcome to the *third* FTC for vector fields. It's the most powerful one because it simplifies your work tremendously. It uses the concept of divergence, which we recall now:

1. Recap: Divergence

Divergence

If $F = \langle P, Q, R \rangle$, then $\text{div}(F) = P_x + Q_y + R_z$

$\text{div}(F)$ measures how much F expands:



Date: Monday, December 6, 2021.

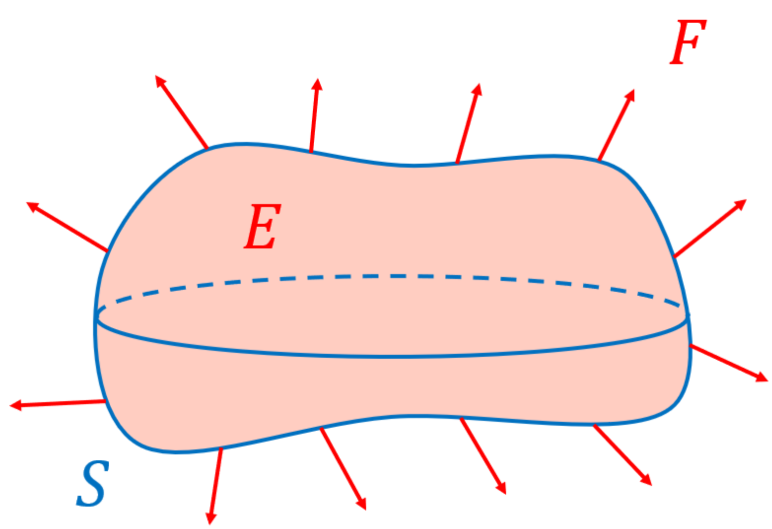
2. The Divergence Theorem

$$\text{Motivation: } \int\int F = \int\int\int F'$$

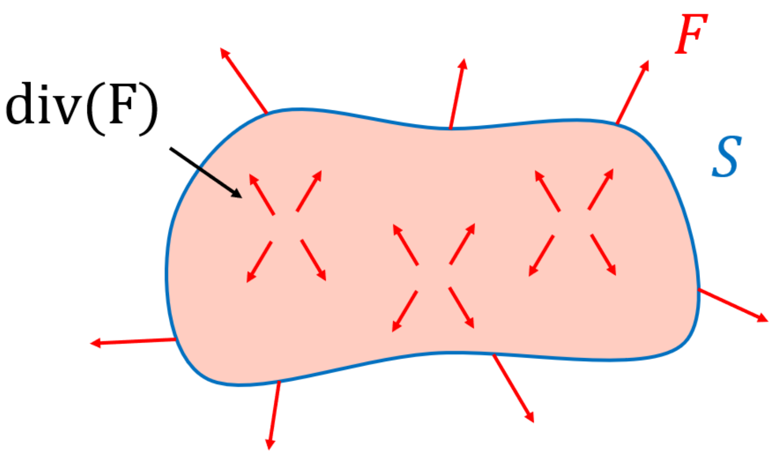
The Divergence Theorem

$$\int\int_S F \cdot d\mathbf{S} = \int\int\int_E \operatorname{div}(F)\, dxdydz$$

Here S is a closed surface and E the region inside S



Interpretation: If you add up all the mini-expansions $\operatorname{div}(F)$ over E, you get the net flux of F over S:



3. Examples

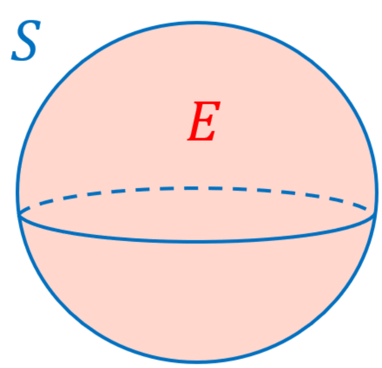
Video: The Divergence Theorem

Example 1:

$$\int\int_S F \cdot d\mathbf{S}$$

$F = \langle 3x, 2y, -z \rangle$ and S : Sphere of Radius 2

Picture:



$$\text{div}(F) = (3x)_x + (2y)_y + (-z)_z = 3 + 2 - 1 = 4$$

$$\begin{aligned}
&\int\int_S F \cdot d\mathbf{S} \\
=&\int\int\int_E \text{div}(F)\, dxdydz \\
=&\int\int\int_E 4\, dxdydz \\
=&4\,\text{Vol}(E) \\
=&4\left(\frac{4}{3}\right)\pi\left(2^3\right) \\
=&\frac{128\pi}{3} \qquad \textbf{WOW}
\end{aligned}$$

Video: The Divergence Theorem

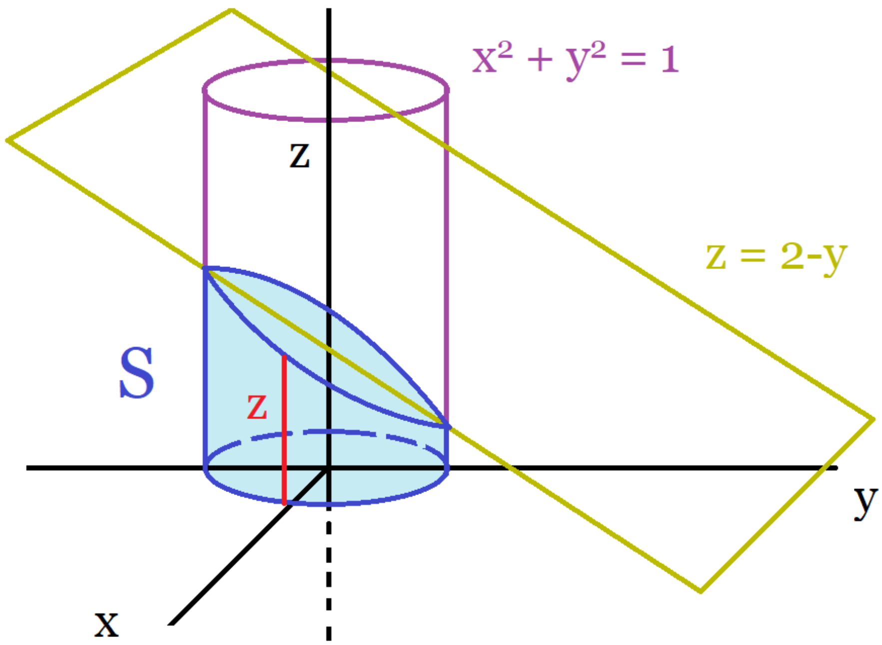
Example 2:

$$\int\int_S F \cdot d\mathbf{S}$$

$$F = \left\langle xy, y^2 + e^{xz^2}, \sin(xy) \right\rangle$$

S : Surface bounded by $x^2 + y^2 = 1$, $z = 0$, and $y + z = 2$

Picture:



Note: Evaluating $\int\int_S F \cdot d\mathbf{S}$ directly is **painful**, you would have to evaluate 3 different surface integrals!

$$
\begin{aligned}
&\int\int_S F \cdot d\mathbf{S} \\
=&\int\int\int_E \mathrm{div}(F)\, dxdydz \\
=&\int\int\int_E (xy)_x + \left(y^2 + e^{xz^2}\right)_y + (\sin(xy))_z \\
=&\int\int\int_E y + 2y + 0 \\
=&\int\int\int_E 3y\, dxdydz
\end{aligned}
$$

Inequalities:

$$\begin{cases} 0 \leq z \leq 2 - y = 2 - r\sin(\theta) 0 \leq \quad r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$\begin{aligned}
&\int\int\int_E 3y \, dxdydz \\
=&\int_0^{2\pi}\int_0^1\int_0^{2-r\sin(\theta)} 3r\sin(\theta)\, r dzdrd\theta \\
=&\int_0^{2\pi}\int_0^1 3r^2\sin(\theta)(2 - r\sin(\theta))drd\theta \\
=&\int_0^{2\pi}\int_0^1 6r^2\sin(\theta) - 3r^3\sin^2(\theta)drd\theta \\
=&\left(\int_0^1 6r^2 dr\right)\int_0^{2\pi}\sin(\theta)d\theta - \left(\int_0^1 3r^3 dr\right)\int_0^{2\pi}\sin^2(\theta)d\theta \\
=&(2)(0) - \frac{3}{4}\int_0^{2\pi}\frac{1}{2} - \frac{1}{2}\cos(2\theta)d\theta \\
=&-\frac{3}{4}\left[\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} \\
=&-\frac{3\pi}{4}
\end{aligned}$$

4. Closing a Surface

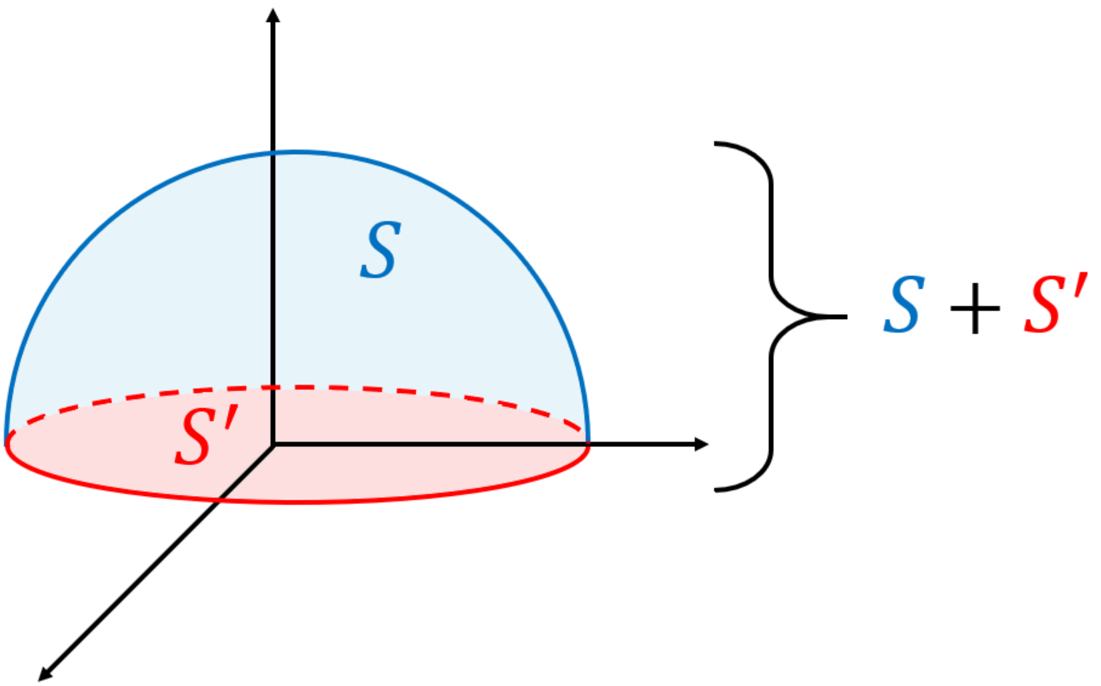
Example 3:

$$\int\int_S F \cdot d\mathbf{S}$$

$$F = \left\langle z^2 x, \left(\frac{1}{3}\right) y^3 + \tan(z), x^2 z + y^2 \right\rangle$$

S : Top half of sphere $x^2 + y^2 + z^2 = 1$ (without bottom)

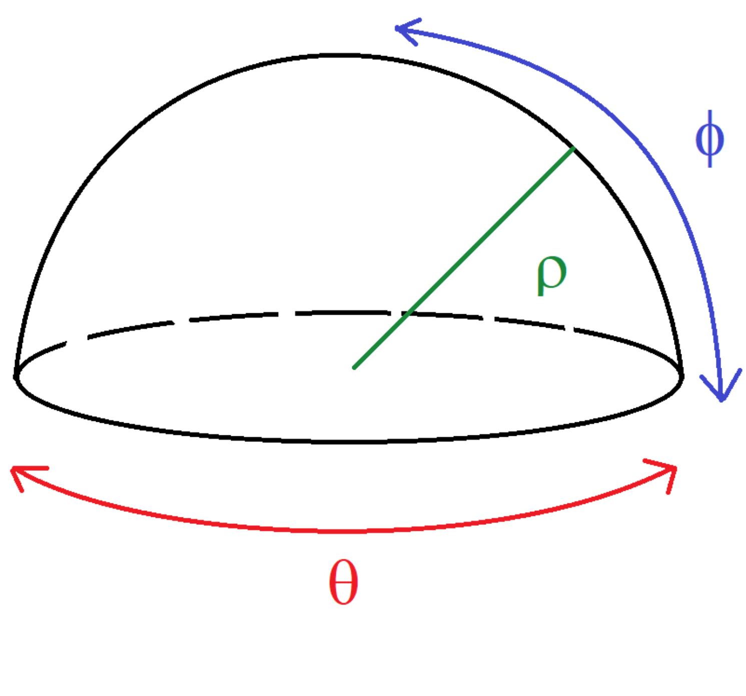
STEP 1: Picture:



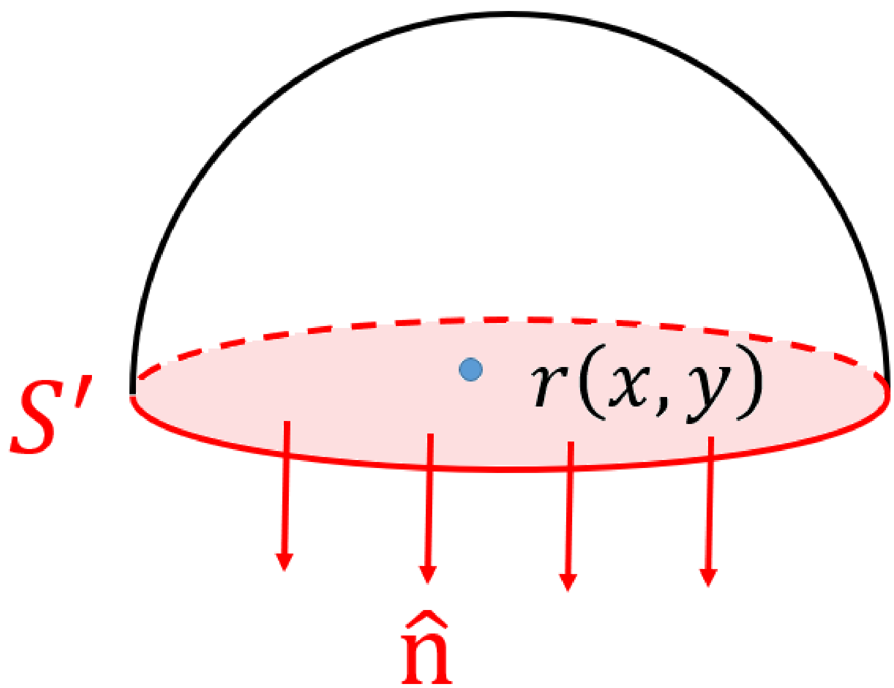
⚠ S is not closed! (doesn't include the bottom lid), so need to close it!

Let S' = bottom disk, then $S + S'$ is closed, so by the Div Thm:

$$
\begin{aligned}
&\int\int_{S+S'} F \cdot d\mathbf{S} \\
=&\int\int\int_E \operatorname{div}(F) dxdydz \\
=&\int\int\int_E \left(z^2x\right)_x + \left(\frac{1}{3}y^3 + \tan(z)\right)_y + \left(x^2z + y^2\right)_z \; dxdydz \\
=&\int\int\int_E z^2 + y^2 + x^2 \; dxdydz \\
=&\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^2 \, \rho^2 \sin(\phi) \; d\rho d\theta d\phi \\
=&2\pi \left(\int_0^{\frac{\pi}{2}} \sin(\phi) d\phi\right) \left(\int_0^1 \rho^4 d\rho\right) \\
=&\frac{2\pi}{5}
\end{aligned}
$$



STEP 2: $\int\int_{S'} F \cdot d\mathbf{S}$ (bottom disk)



Make sure that $\hat{n}$ points **downwards** here (because want outward orientation)

(1) **Parametrize:** $r(x, y) = \langle x, y, 0\rangle$ (or use polar coordinates)

(2) **Partial Derivatives:** $r_x = \langle 1, 0, 0\rangle$, $r_y = \langle 0, 1, 0\rangle$

(3) **Normal Vector**

$$\hat{n} = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, 1\rangle$$

Since we want $\hat{n}$ to point downwards, we choose $\hat{n} = \langle 0, 0, -1 \rangle$

(4)

$$\begin{aligned}
&\int\int_{S'} F \cdot d\mathbf{S} \\
=&\int\int_{D} F \cdot \hat{n}\, dxdy \\
=&\int\int_{D} \left\langle 0x^2 + \frac{1}{3}y^3 + \tan(0), x^2(0) + y^2 \right\rangle \cdot \langle 0, 0, -1\rangle\, dxdy \\
=&\int\int_{D} -y^2 dxdy \qquad D: \text{ Disk of radius 1} \\
=&\int_0^{2\pi}\int_0^1 -r^2\sin^2(\theta) r dr d\theta \\
=&\left(\int_0^1 -r^3 dr\right)\left(\int_0^{2\pi}\sin^2(\theta)d\theta\right) \\
=&\left(-\frac{1}{4}\right)\int_0^{2\pi}\frac{1}{2} - \frac{1}{2}\cos(2\theta)d\theta \\
=&-\frac{1}{4}\left[\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} \\
=&-\frac{1}{4}(\pi) \\
=&-\frac{\pi}{4}
\end{aligned}$$

STEP 3: $\int\int_S F \cdot d\mathbf{S}$ (Sphere part)

$$\int\int_{S+S'} F \cdot d\mathbf{S} = \int\int_S F \cdot d\mathbf{S} + \int\int_{S'} F \cdot d\mathbf{S}$$

$$\begin{aligned}\int\int_S F\cdot d\mathbf{S} &= \int\int_{S+S'} F\cdot d\mathbf{S} - \int\int_{S'} F\cdot d\mathbf{S}\\ &= \frac{2\pi}{5} - \left(-\frac{\pi}{4}\right)\\ &= \frac{13\pi}{20}\end{aligned}$$

5. Stokes' Theorem (if time permits)

Video: Stokes' Theorem

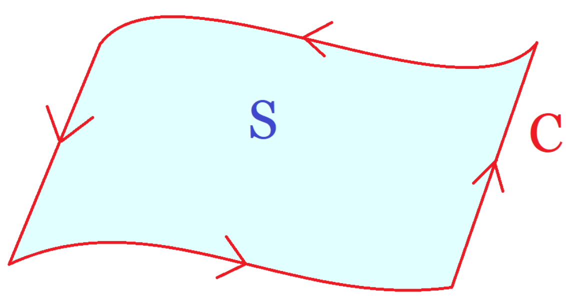
Let's get Stoked for our fourth and final FTC for vector fields: Stokes' Theorem!

$$\text{Motivation: } \int\int F' = \int F$$

Stokes' Theorem

If S is surface with boundary curve C, then:

$$\int\int_S \operatorname{curl}(F)\cdot d\mathbf{S} = \int_C F\cdot dr$$



Note: Here I'll just give a quick example. I'll remind you what curl is next time, since we won't need it for now.

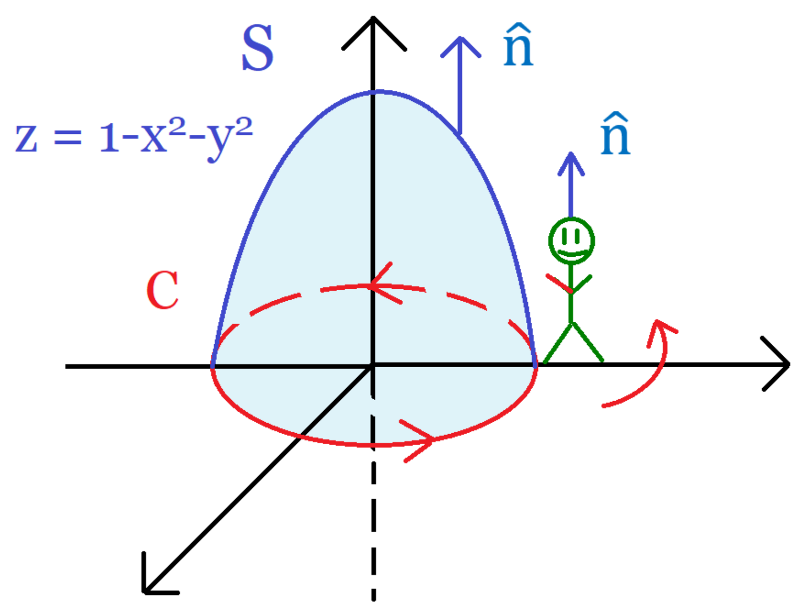
Example 4:

Evaluate $\int\int_S \operatorname{curl}(F) \cdot d\mathbf{S}$

$F = \langle xz, y^2, xy \rangle$

S is the paraboloid $z = 1 - x^2 - y^2$ above the $xy-$plane

STEP 1: Picture:



⚠ Orientation matters! If you're walking on C with your head in the direction of $\hat{n}$, then S should be to your **LEFT**

Mnemonic: WALK LEFT

So here C is counterclockwise (most of the time it is)

STEP 2: By Stokes:

$$\int\int_S \text{curl}(F) \cdot d\mathbf{S} = \int_C F \cdot dr$$

C is a circle of radius 1 ($z = 1 - x^2 - y^2$ and $z = 0$ gives $x^2 + y^2 = 1$)

STEP 3: Parametrize C: $r(t) = \langle \cos(t), \sin(t), 0 \rangle, 0 \leq t \leq 2\pi$

$$\begin{aligned}
&\int_C F \cdot dr \\
=&\int_0^{2\pi} F(r(t)) \cdot r'(t) dt \\
=&\int_0^{2\pi} \underbrace{\left\langle \cos(t)(0), \sin^2(t), \cos(t)\sin(t) \right\rangle}_{\langle xz, y^2, xy \rangle} \cdot \underbrace{\langle -\sin(t), \cos(t), 0 \rangle}_{r'(t)} dt \\
=&\int_0^{2\pi} \sin^2(t)\cos(t) dt \\
=&\left[\frac{1}{3}\sin^3(t)\right]_0^{2\pi} \\
=&0
\end{aligned}$$

Note: For a more interesting version of this problem, check out:

Video: Integral over a Barrel