LECTURE 42: STOKES' THEOREM

There's a saying in German that says "Everything has an end, except for a sausage, which has two"[1] And with this, I would like to welcome you to the final lecture of your final calculus course!

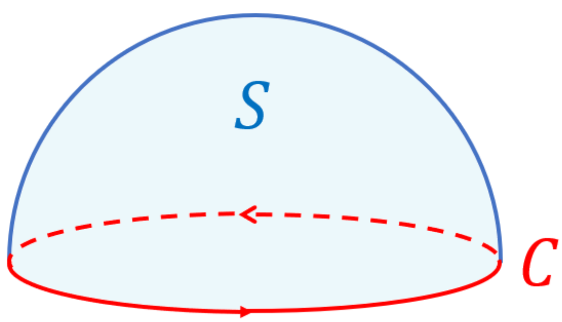
1. Stokes' Theorem

Previously on *I'm so Stoked*, we learned about an amazing theorem that relates line integrals with surface integrals.

Stokes' Theorem

Let S be a surface with boundary C, then:

$$\int_C F \cdot dr = \int\int_S \mathrm{curl}(F) \cdot d\mathbf{S}$$



Last time: We used Stokes to calculate $\int\int_S \mathrm{curl}(F)$, but today we'll use it to calculate $\int_C F \cdot dr$

Date: Wednesday, December 8, 2021.

[1] Alles hat ein Ende, nur die Wurst, die hat zwei

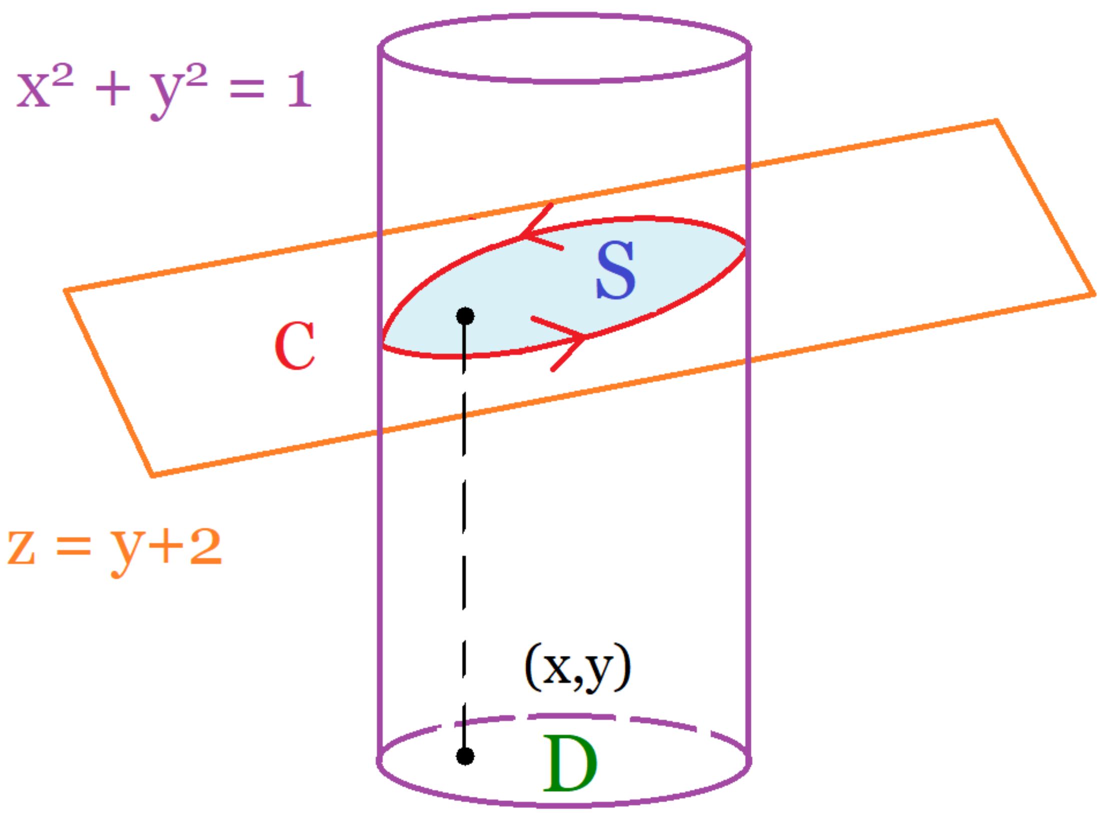
2. A Cylindrical Example

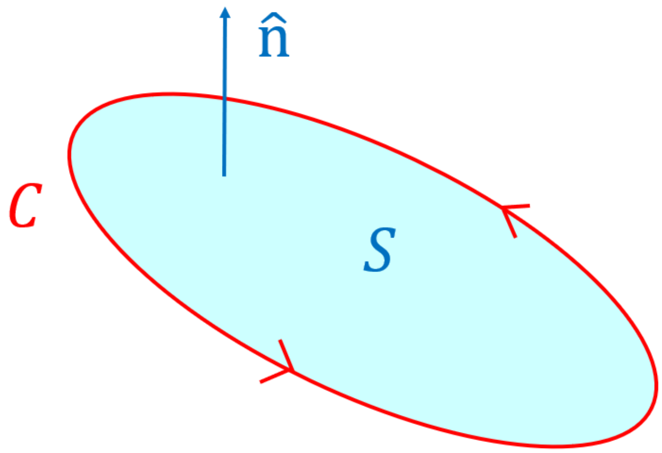
Video: Stokes' Theorem

Example 1:

Evaluate $\int_C F \cdot dr$, where $F = \langle xy, yz, xz \rangle$ and C is the curve of intersection of $z = y + 2$ and $x^2 + y^2 = 1$ (in the counterclockwise direction)

STEP 1: Picture





STEP 2: By Stokes, we have:

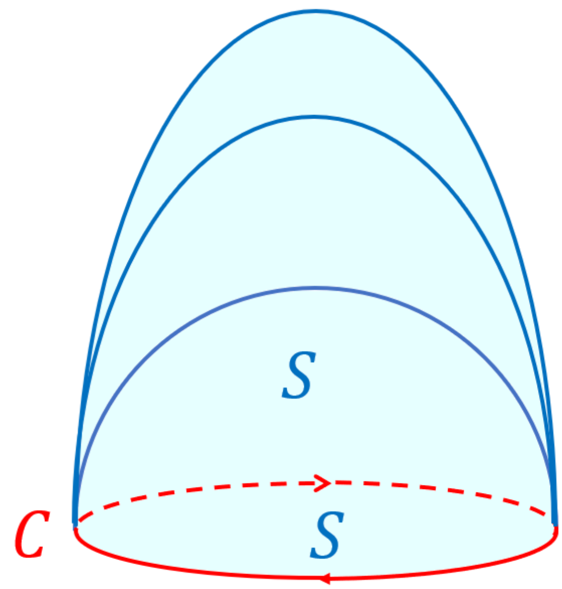
$$\int_C F \cdot dr = \int\int_S \text{curl}(F) \cdot d\mathbf{S}$$

$$\begin{aligned}
\text{curl}(F) =& \nabla \times F \\
=& \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} \\
=& \left\langle \frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(yz), -\frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial z}(xy), \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy) \right\rangle \\
=& \langle -y, -z, -x \rangle \qquad \textit{EASIER}
\end{aligned}$$

STEP 3: What is S?

Stokes' Miracle

S can be anything we want, as long as the boundary of S is C



This is because $\int_C F \cdot dr$ only depends on C, not on S at all.

Analogy: Think of S as a hot air balloon, then as long as the string C is still the same, it doesn't matter what the shape of the balloon is.

Easiest choice: Let S be the interior (inside) of C.

Parametrize S:

$$r(x,y) = \left\langle x, y, \underbrace{y+2}_{z} \right\rangle$$

$$r_x = \langle 1, 0, 0 \rangle$$

$$r_y = \langle 0, 1, 1 \rangle$$

$$\hat{n} = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \langle 0, -1, 1 \rangle$$

(Mentally check WALK Left, that is, S is to the left of the curve C)

STEP 4:

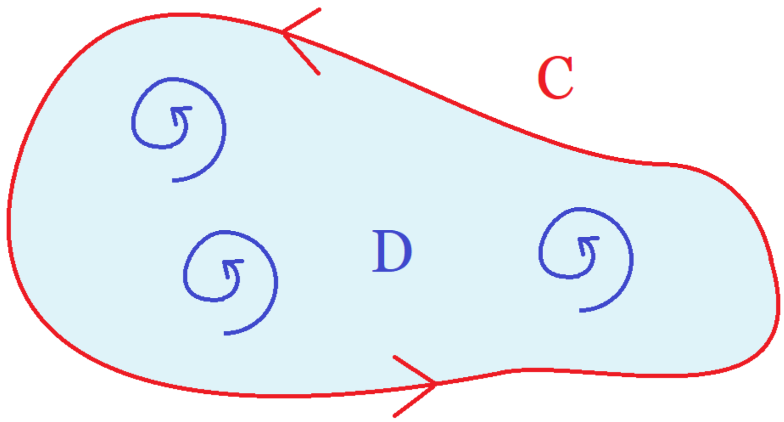
$$\begin{aligned}
\int_C F \cdot dr &= \int\int_S \operatorname{curl} F \cdot d\mathbf{S} \\
&= \int\int_D \underbrace{\langle -y, -(y+2), -x\rangle}_{\langle -y,-z,-x\rangle} \cdot \underbrace{\langle 0,-1,1\rangle}_{\hat{n}} dxdy \\
&= \int\int_D y + 2 - x dxdy \qquad \text{(D is a disk of radius 1)} \\
&= \int_0^{2\pi}\int_0^1 \left(r\sin(\theta) + 2 - r\cos(\theta)\right) rdrd\theta \\
&= \int_0^{2\pi}\int_0^1 r^2\sin(\theta) + 2r - r^2\cos(\theta) drd\theta \\
&= \left(\int_0^1 r^2 dr\right)\left(\int_0^{2\pi}\sin(\theta)d\theta\right) + 2\pi\left(\int_0^1 2rdr\right) \\
&- \left(\int_0^1 r^2 dr\right)\left(\int_0^{2\pi}\cos(\theta)d\theta\right) \\
&= 0 + 2\pi(1) - 0 \\
&= 2\pi
\end{aligned}$$

3. Intuition

Stokes is nothing other than a $3D$ analog of Green:

Recall: Green's Theorem

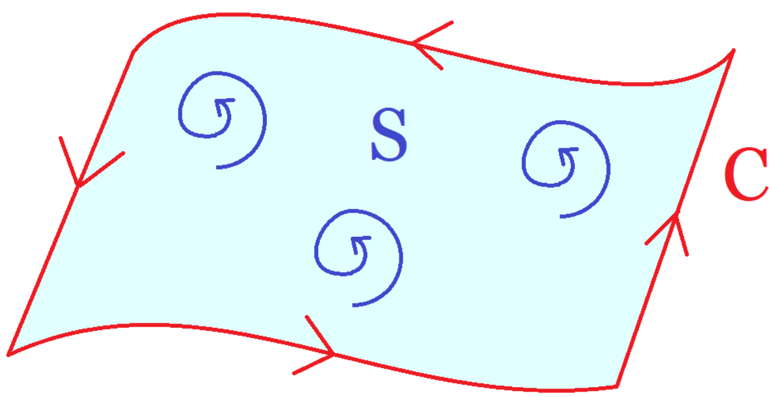
$$\underbrace{\int_C F \cdot dr}_{\text{Macro Rotation}} = \underbrace{\int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy}_{\text{Sum of Micro-Rotations}}$$



Stokes says the same thing, but in 3D:

Stokes:

$$\underbrace{\int_C F \cdot dr}_{\text{Macro Circulation}} = \underbrace{\int\int_S \text{curl}(F) \cdot d\mathbf{S}}_{\text{Sum of Micro Rotations}}$$



Analogy: Suppose you want to count the number of cars in a parking lot. You could either walk around the lot and count all the cars ($\int_C F \cdot dr$) or you could walk inside the lot and count how many cars go in and out of the lot ($\int\int_S F \cdot d\mathbf{S}$)

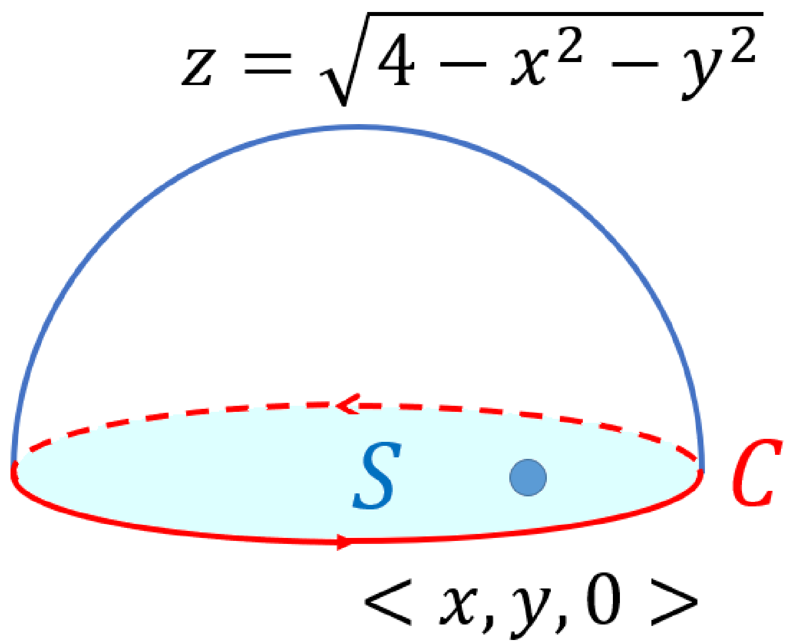
4. A Spherical Example

Video: More Stokes' Theorem

Example 2:

Calculate $\int_C F \cdot dr$ where $F = \left\langle x + y^2, y + z^2, z + x^2 \right\rangle$ and C is the boundary of the hemisphere $z = \sqrt{4 - x^2 - y^2}$ in the $xy-$plane (oriented counterclockwise)

STEP 1: Picture



STEP 2: By Stokes:

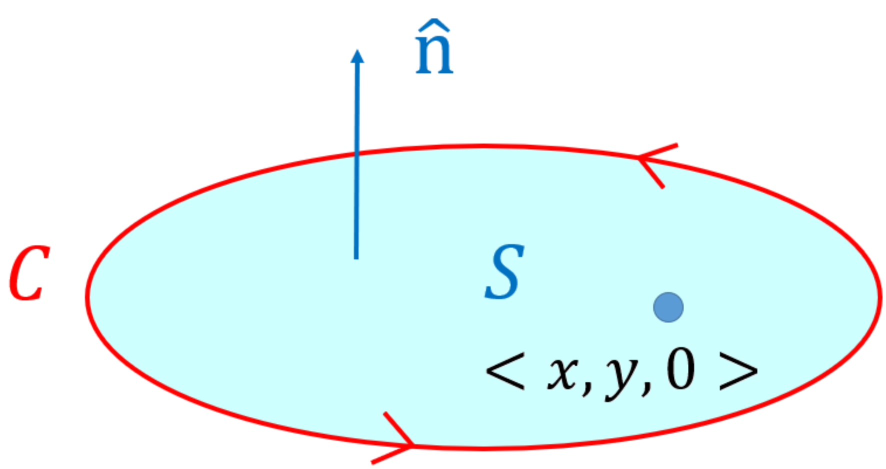
$$\int_C F \cdot dr = \int\int_S \text{curl}(F) \cdot d\mathbf{S}$$

$$\begin{aligned}
\text{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y^2 & y + z^2 & z + x^2 \end{vmatrix} \\
&= \left\langle \frac{\partial}{\partial y}(z + x^2) - \frac{\partial}{\partial z}(y + z^2), -\frac{\partial}{\partial x}(z + x^2) + \frac{\partial}{\partial z}(x + y^2), \right. \\
&\quad \left. \frac{\partial}{\partial x}(y + z^2) - \frac{\partial}{\partial y}(x + y^2) \right\rangle \\
&= \langle -2z, -2x, -2y \rangle \qquad \textit{EASIER}
\end{aligned}$$

STEP 3: What is S?

Here C is a circle of radius 2, because if you let $z = 0$ in $z = \sqrt{4 - x^2 - y^2}$ you get $x^2 + y^2 = 4$

So let S be the **DISK** of radius 2 in the $xy-$plane!



Parametrize: $r(x,y) = \langle x, y, 0 \rangle$

Slopes: $r_x = \langle 1, 0, 0 \rangle, r_y = \langle 0, 1, 0 \rangle$

Normal Vector:

$$\hat{n} = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, 1 \rangle$$

STEP 4:

$$\begin{aligned} \int_C F \cdot dr &= \int\int_S \mathrm{curl}(F) \cdot d\mathbf{S} \\ &= \int\int_D \underbrace{\langle 0, -2x, -2y \rangle}_{\langle -2z, -2x, -2y \rangle} \cdot \underbrace{\langle 0, 0, 1 \rangle}_{\hat{n}} dxdy \\ &= \int\int_D -2y dxdy \qquad \text{(D is a disk of radius 2)} \\ &= \int_0^{2\pi} \int_0^2 -2r\sin(\theta) r dr d\theta \\ &= \left(\int_0^2 -2r^2 dr\right)\left(\int_0^{2\pi} \sin(\theta) d\theta\right) \\ &= 0 \end{aligned}$$

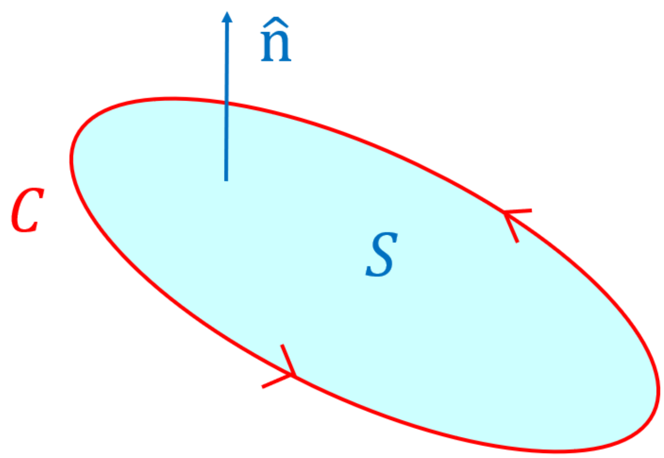
5. Concluding Remarks

Remark 1:

Recall

IF $F = \langle P, Q, R \rangle$ is conservative, **THEN** $\mathrm{curl}(F) = \langle 0, 0, 0 \rangle$

Now if $\text{curl}(F) = \langle 0, 0, 0 \rangle$ (and no holes), then for any closed C,

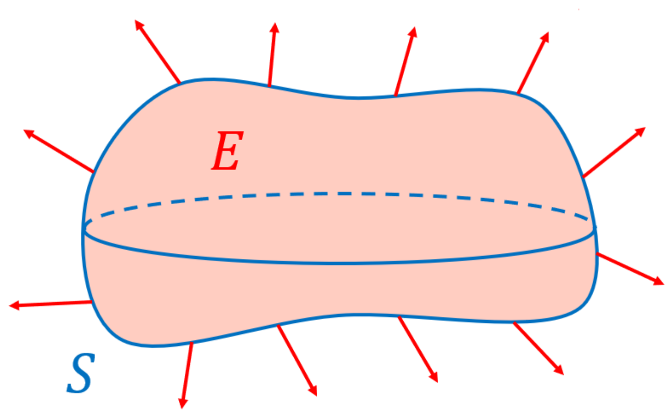


$$\int_C F \cdot dr = \int\int_S \underbrace{\text{curl}(F)}_{\langle 0,0,0 \rangle} \cdot d\mathbf{S} = 0$$

So F is conservative by neat fact from 16.3

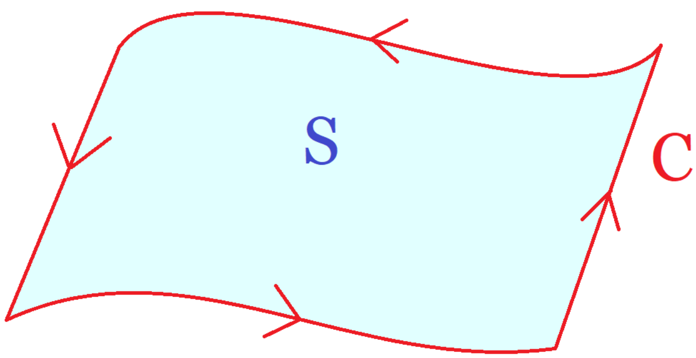
Hence we have: F conservative $\Leftrightarrow \text{curl}(F) = \langle 0, 0, 0 \rangle$

Remark 2: Important: If S is **closed**, then:



$$\int\int_S \text{curl}(F) \cdot d\mathbf{S} \overset{DIV}{=} \int\int\int_E \underbrace{\text{div}(\text{curl}(F))}_{0} = 0$$

Point: Stokes in only interesting for *open* surfaces, where there is a boundary curve C



Remark 3: Stokes FAILS when there is a hole.

Cohomology Theory: How badly does it fail? Gives us interesting information about the *topology* of the domain, think spheres vs donuts

Alright!!! This is officially the end of calculus adventure! Thank you for flying Peyam Airlines, it's been a pleasure having you on board, and I wish you a safe onward journey!

The End