LECTURE 5: LIMITS OF SEQUENCES (II)

Today is all about practice with the definition of the limit

1. Example 1: The Basics

Video: Limit Example 1: The Basics

Example 1:

$$\text{Show: } \lim_{n\to\infty} 3 - \frac{1}{n^2} = 3$$

Show: For all $\epsilon > 0$ there is $N > 0$ such that if $n > N$ then:

$$|s_n - s| < \epsilon$$

STEP 1: Find N

Note: This step is scratchwork and is technically not part of your proof. The goal here is to find N and you do that by solving for n in $|s_n - s| < \epsilon$:

$$|s_n - s| = \left| \left(3 - \frac{1}{n^2}\right) - 3 \right| = \left| -\frac{1}{n^2} \right| = \frac{1}{n^2} < \epsilon$$

Which gives $n^2 > \frac{1}{\epsilon} \Rightarrow n > \sqrt{\frac{1}{\epsilon}} = \frac{1}{\sqrt{\epsilon}}$.

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Therefore let $N = \frac{1}{\sqrt{\epsilon}}$ (Note that N is not necessarily an integer)

STEP 2: Our actual proof:

Let $\epsilon > 0$ be given and let $N = \frac{1}{\sqrt{\epsilon}}$. Then if $n > N = \frac{1}{\sqrt{\epsilon}}$, we have:

$$|s_n - s| = \left| 3 - \frac{1}{n^2} - 3 \right| = \left| \frac{-1}{n^2} \right| = \frac{1}{n^2}$$

But if $n > \sqrt{\frac{1}{\epsilon}}$, then $n^2 > \frac{1}{\epsilon}$, so $\frac{1}{n^2} < \epsilon$, and hence

$$|s_n - s| = \frac{1}{n^2} < \epsilon \checkmark$$

Therefore $\lim_{n\to\infty} 3 - \frac{1}{n^2} = 3$ $\square$

2. Example 2: Simple Fraction

Video: Limit Example 2: Simple Fraction

Example 2:

$$\lim_{n\to\infty} \frac{2n+4}{4n+5} = \frac{1}{2}$$

Note: Intuitively this should be true because

$$\frac{2n+4}{4n+5} \approx \frac{2n}{4n} = \frac{2}{4} = \frac{1}{2}$$

Show for all $\epsilon > 0$ there is N such that if $n > N$, then $|s_n - s| < \epsilon$

STEP 1: Find N

$$
\begin{aligned}
|s_n - s| &= \left|\frac{2n+4}{4n+5} - \frac{1}{2}\right| = \left|\frac{(2n+4)(2) - (4n+5)}{2(4n+5)}\right| \\
&= \left|\frac{4n+8-4n-5}{2(4n+5)}\right| = \left|\underbrace{\frac{3}{2(4n+5)}}_{>0}\right| = \frac{3}{2(4n+5)} < \epsilon
\end{aligned}
$$

$$
\begin{aligned}
\text{However,} \qquad \frac{3}{2(4n+5)} &< \epsilon \\
\Rightarrow \frac{1}{4n+5} &< \frac{2\epsilon}{3} \\
\Rightarrow 4n+5 &> \frac{3}{2\epsilon} \\
\Rightarrow 4n &> \frac{3}{2\epsilon} - 5 \\
\Rightarrow n &> \frac{3}{8\epsilon} - \frac{5}{4}
\end{aligned}
$$

This suggests to let $\boxed{N = \frac{3}{8\epsilon} - \frac{5}{4}}$.

STEP 2: Let $\epsilon > 0$ be given, let $N = \frac{3}{8\epsilon} - \frac{5}{4}$, then if $n > N$, we have

$$
|s_n - s| = \frac{3}{2(4n+5)}
$$

But if $n > N$, then

$$
4n + 5 > 4\left(\frac{3}{8\epsilon} - \frac{5}{4}\right) + 5 = \frac{3}{2\epsilon} - 5 + 5 = \frac{3}{2\epsilon}
$$

Therefore $\frac{1}{4n+5} < \frac{2\epsilon}{3}$, and so

$$
|s_n - s| = \frac{3}{2(4n+5)} < \left(\frac{3}{2}\right)\left(\frac{2\epsilon}{3}\right) = \epsilon \checkmark
$$

Therefore $\lim_{n\to\infty} \frac{2n+4}{4n+5} = \frac{1}{2}$

IMPORTANT: Your absolutely **HAVE** to write down both steps, even if it seems repetitive (Because Step 1 is just scratch work to find N, but in step 2, you're proving that your N works). Otherwise you'll lose points on the exam.

3. Example 3: A Complex Fraction

Video: Limit Example 3: A Complex Fraction

Example 3:

$$\lim_{n\to\infty} \frac{2n^3+3n}{n^3-2} = 2$$

Intuitively this is true because $\frac{2n^3+3n}{n^3-2} \approx \frac{2n^3}{n^3} = 2$

STEP 1:

$$\begin{aligned}
|s_n - s| &= \left| \frac{2n^3+3n}{n^3-2} - 2 \right| \\
&= \left| \frac{2n^3+3n-2\left(n^3-2\right)}{n^3-2} \right| \\
&= \left| \frac{3n+4}{n^3-2} \right| \\
&= \frac{3n+4}{n^3-2} \quad \breve{\text{if}}\ n^3-2>0 \\
&< \epsilon
\end{aligned}$$

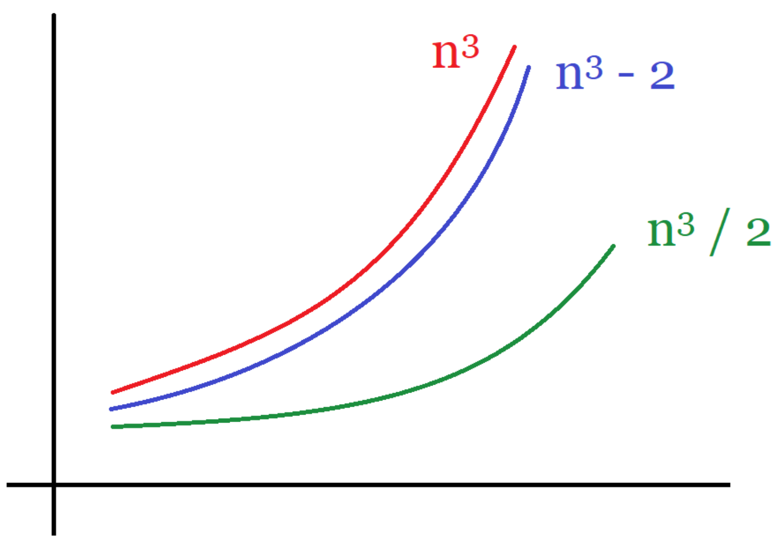
Note: $n^3 - 2 > 0 \Rightarrow n > \sqrt[3]{2}$, so we ăt least need $\boxed{n > \sqrt[3]{2}}$.

Unlike the previous problem, here the fraction is trickier. We need to analyze the numerator and denominator separately.

Numerator: We want $3n + 4 <$ some number. But notice that if $\boxed{n > 1}$, then $4n > 4$, so $4 < 4n$, so $3n + 4 < 3n + 4n = 7n$.

Hence $3n + 4 < 7n$

Denominator: We want $n^3 - 2 >$ some large number (because we'll take reciprocals). The idea is that, even though $n^3 - 2 < n^3$, we still have $n^3 - 2 > \frac{n^3}{2}$ for large n,[1] as in the picture below:



$$\text{But } n^3 - 2 > \frac{n^3}{2} \Rightarrow \left(1 - \frac{1}{2}\right) n^3 > 2 \Rightarrow \frac{n^3}{2} > 2 \Rightarrow n^3 > 4 \Rightarrow \boxed{n > \sqrt[3]{4}}$$

[1] There's nothing special about the factor $\frac{1}{2}$, we could have also done $\frac{n^3}{3}$, that's completely fine

Hence $n^3 - 2 > \frac{n^3}{2}$ so $\frac{1}{n^3-2} < \frac{1}{\frac{n^3}{2}}$.

Fraction: Therefore, if both of the above conditions hold, we get:

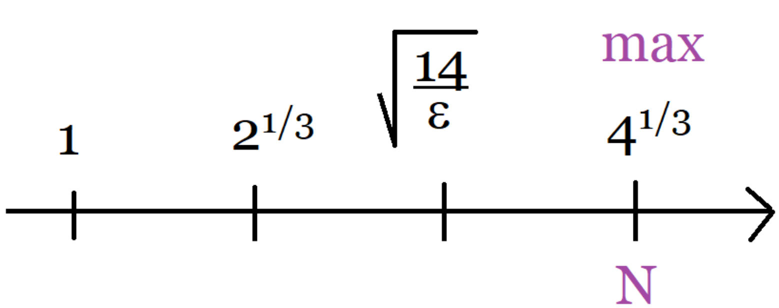
$$\frac{3n+4}{n^3-2} < \frac{7n}{\frac{n^3}{2}} = \frac{14n}{n^3} = \frac{14}{n^2}$$

And therefore

$$\frac{14}{n^2} < \epsilon \Rightarrow \frac{n^2}{14} > \frac{1}{\epsilon} \Rightarrow n^2 > \frac{14}{\epsilon} \Rightarrow n > \sqrt{\frac{14}{\epsilon}}$$

This suggests to let $N = \sqrt{\frac{14}{\epsilon}}$, but since we also need $n > \sqrt[3]{2}$, $n > 1$ and $n > \sqrt[3]{4}$ (see boxed numbers above), N actually needs to be the **larger** one of those 4 numbers, in other words N is the *max* of $\sqrt[3]{2}, 1, \sqrt[3]{4}, \sqrt{\frac{14}{\epsilon}}$

STEP 2: Let $\epsilon > 0$ and let $N = \max\left\{\sqrt[3]{2}, 1, \sqrt[3]{4}, \sqrt{\frac{14}{\epsilon}}\right\} = \max\left\{\sqrt[3]{4}, \sqrt{\frac{14}{\epsilon}}\right\}$ (since $\sqrt[3]{4} > 1$ and $\sqrt[3]{4} > \sqrt[3]{2}$)



Then if $n > N$, we have:

$$\begin{aligned}
|s_n - s| &= \left|\frac{2n^3+3n}{n^3-2} - 2\right| \\
&= \left|\frac{3n+4}{n^3-2}\right| \\
&= \frac{3n+4}{n^3-2} && \text{Since } n > \sqrt[3]{2}\text{, so } n^3 - 2 > 0 \\
&< \frac{7n}{n^3-2} && \text{Since } n > 1 \text{ so } 3n+4 < 3n+4n = 7n \\
&= \frac{7n}{\frac{n^3}{2}} && \text{Since } n > \sqrt[3]{4} \text{ so } n^3 - 2 > \frac{n^3}{2} \\
&= \frac{14}{n^2}
\end{aligned}$$

$$\text{But } n > \sqrt{\frac{14}{\epsilon}} \Rightarrow n^2 > \frac{14}{\epsilon} \Rightarrow \frac{1}{n^2} < \frac{\epsilon}{14}$$

Therefore:

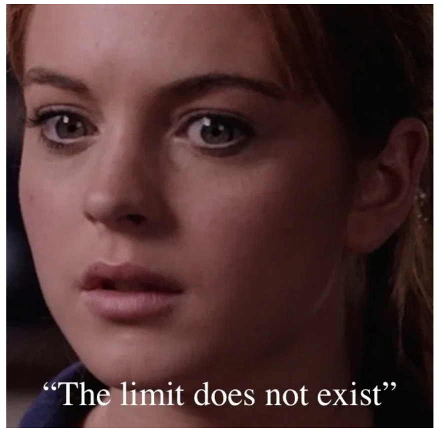
$$|s_n - s| = \frac{14}{n^2} < 14\left(\frac{\epsilon}{14}\right) = \epsilon ✓$$

Hence $\lim_{n\to\infty} \frac{2n^3+3n}{n^3-2} = 2$ □

4. Example 4: The Limit Does Not Exist

Video: Limit Example 4: The Limit Does Not Exist

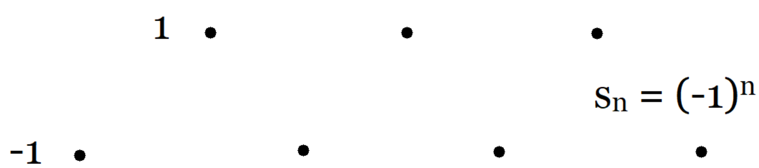
In this example, we'll see what happens when (to quote Mean Girls)



Example 4:

Show that the following limit does not exist:

$$\lim_{n\to\infty} (-1)^n$$



This is hard to do directly, so let's do it by contradiction!

(1) Suppose $\lim_{n\to\infty} s_n = s$ for some s.

Let $\epsilon > 0$ be TBA.

Then there is $N > 0$ such that if $n > N$ then:

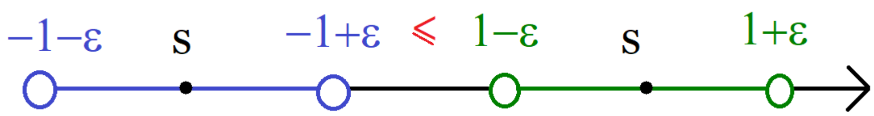
$$|(-1)^n - s| < \epsilon$$

(2) If n is even, this becomes:

$$|1 - s| = |s - 1| < \epsilon \Rightarrow -\epsilon < s - 1 < \epsilon \Rightarrow 1 - \epsilon < s < 1 + \epsilon$$

(3) If n is odd, then we get:

$$|-1 - s| = |s + 1| < \epsilon \Rightarrow -\epsilon < s + 1 < \epsilon \Rightarrow -1 - \epsilon < s < -1 + \epsilon$$



(4) Finally, choose $\epsilon > 0$ such that $-1 + \epsilon \leq 1 - \epsilon$ (for instance $\epsilon = 1$ works). Then we get the contradiction:

$$s < -1 + \epsilon \leq 1 - \epsilon < s \Rightarrow\Leftarrow$$

Therefore $\lim_{n\to\infty}(-1)^n$ does not exist.

5. Example 5: Square roots

Video: Limit Example 5: Square Roots

Let's continue our practice with limits, this time with square roots!

Example 5:

Show that if $s_n \geq 0$ for all n and $\lim_{n\to\infty} s_n = s$, then

$$\lim_{n\to\infty} \sqrt{s_n} = \sqrt{s}$$

(This will later show that $f(x) = \sqrt{x}$ is continuous)

Note: In this proof, assume $s > 0$. The case $s = 0$ can be dealt with separately (see problem 3 in section 8).

Show for all $\epsilon > 0$ there is N such that if $n > N$, then $\left|\sqrt{s_n} - \sqrt{s}\right| < \epsilon$.

STEP 1: Scratch work

Just like in Calculus, it's useful to multiply $\sqrt{s_n} - \sqrt{s}$ by its conjugate form $\frac{\sqrt{s_n}+\sqrt{s}}{\sqrt{s_n}+\sqrt{s}}$:

$$\begin{aligned}
\left|\sqrt{s_n} - \sqrt{s}\right| &= \left|\left(\sqrt{s_n} - \sqrt{s}\right)\left(\frac{\sqrt{s_n}+\sqrt{s}}{\sqrt{s_n}+\sqrt{s}}\right)\right| \\
&= \left|\frac{(\sqrt{s_n})^2 - (\sqrt{s})^2}{\sqrt{s_n}+\sqrt{s}}\right| \qquad (A-B)(A+B) = A^2 - B^2 \\
&= \left|\frac{s_n - s}{\sqrt{s_n}+\sqrt{s}}\right| \\
&= \frac{|s_n - s|}{\sqrt{s_n}+\sqrt{s}}
\end{aligned}$$

Now the numerator is small (by assumption) and for the denominator, notice that $\sqrt{s_n} + \sqrt{s} \geq \sqrt{s}$ (doesn't depend on n), hence

$$\frac{|s_n - s|}{\sqrt{s_n}+\sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \epsilon \Rightarrow |s_n - s| < \left(\sqrt{s}\right)\epsilon$$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, then there is N such that if $n > N$, then $|s_n - s| < (\sqrt{s})\,\epsilon$.

But then, for that same N, if $n > N$, we get:

$$\left|\sqrt{s_n} - \sqrt{s}\right| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\left(\sqrt{s}\right)\epsilon}{\sqrt{s}} = \epsilon\checkmark$$

Hence $\lim_{n\to\infty} \sqrt{s_n} = \sqrt{s}$ $\square$