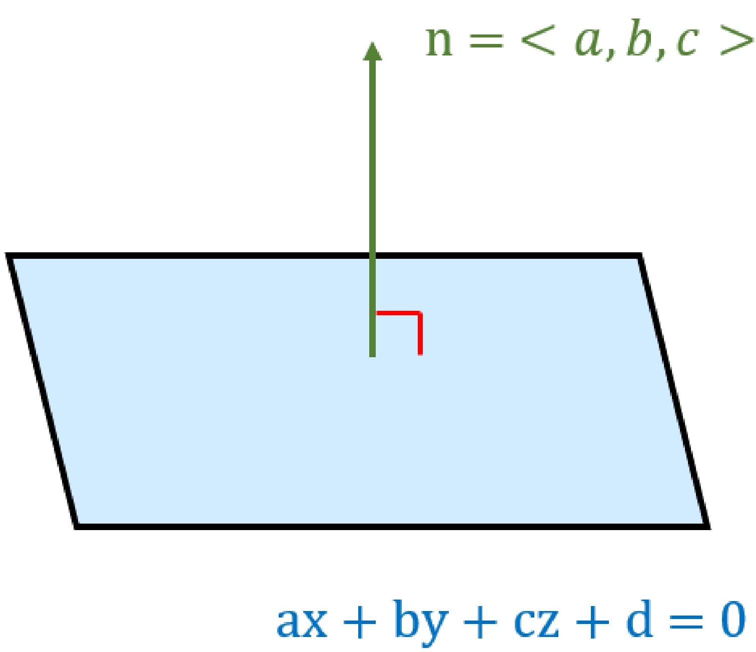
LECTURE 5: FUN WITH PLANES

What is today's lecture about? Is it a bird? Is it a plane? Well yes, it's about planes!

1. Planes

Equation of a plane:

A **plane** has equation $ax + by + cz + d = 0$



Seems simple, but it *the* most important object in multivariable calculus, for reasons you'll see later in the course.

Date: Wednesday, September 8, 2021.

Just like for lines, where we had a direction vector, for planes we have a *normal* vector:

Definition:

$\mathbf{n} = \langle a, b, c \rangle$ is the **normal vector** to the plane $ax+by+cz+d=0$

Normal is a synonym for perpendicular, and in fact:

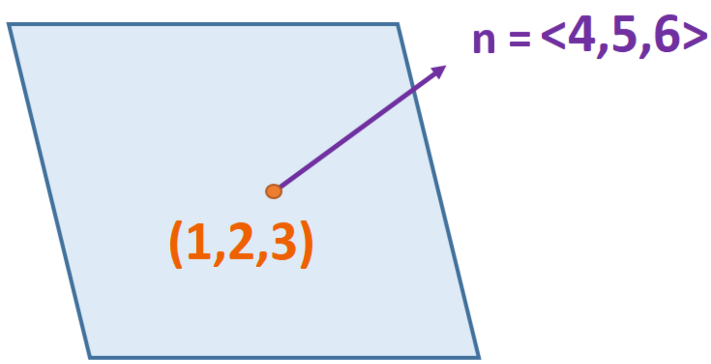
Fact:

$\mathbf{n} = \langle a, b, c \rangle$ is always perpendicular to $ax+by+cz+d=0$

And it's *precisely* this fact that allows us to find equations of planes.

Example 1:

Find the equation of the plane going through $(1, 2, 3)$ and with normal vector $\mathbf{n} = \langle 4, 5, 6 \rangle$



Fact:

The plane has equation

$$4(x-1)+5(y-2)+6(z-3)=0$$

Can be rewritten as $4x + 5y + 6z - 32 = 0$, but ok to leave it like that

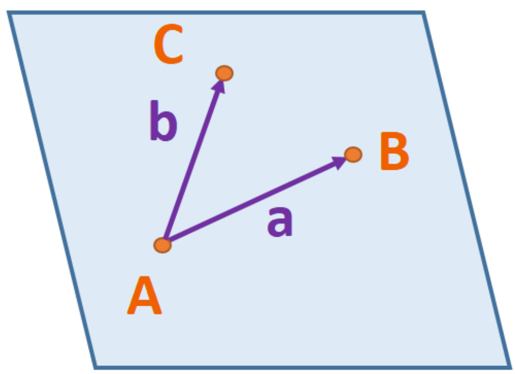
This says that, in order to find a plane, all we need is a **point** and a **normal vector**.

2. Properties

Example 2: (Good Quiz/Exam Question)

Find the equation of the plane containing the points

$$A = (2,1,2), B = (-3,8,6), C = (-2,-3,1)$$



(1) **Point:** $A = (2,1,2)$ (B or C would work as well)

(2) **Normal Vector**

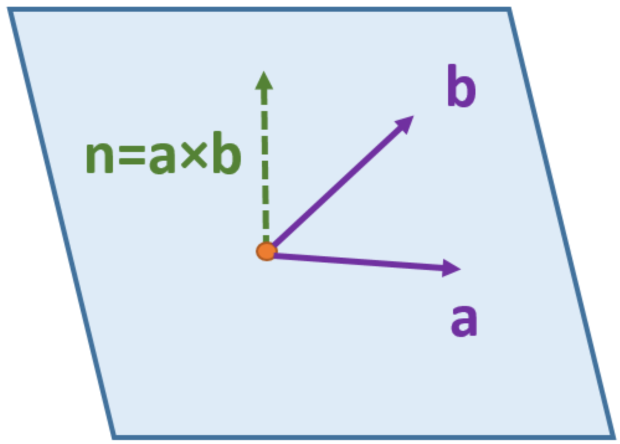
(i) Notice that the following vectors **a** and **b** are on the plane:

$$\mathbf{a} = \overrightarrow{AB} = \langle -3-2, 8-1, 6-2 \rangle = \langle -5, 7, 4 \rangle$$
$$\mathbf{b} = \overrightarrow{AC} = \langle -2-2, -3-1, 1-2 \rangle = \langle -4, -4, -1 \rangle$$

(ii) Find a vector perpendicular to **a** and **b** (and hence to the plane)

Recall:

$\mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b**



$$\begin{aligned}
\mathbf{n} =& \mathbf{a} \times \mathbf{b} \\
=& \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 7 & 4 \\ -4 & -4 & -1 \end{vmatrix} \\
=& \begin{vmatrix} 7 & 4 \\ -4 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -5 & 4 \\ -4 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -5 & 7 \\ -4 & -4 \end{vmatrix} \mathbf{k} \\
=& (-7+16)\mathbf{i} - (5+16)\mathbf{j} + (20+28)\mathbf{k} \\
=& \langle 9, -21, 48 \rangle
\end{aligned}$$

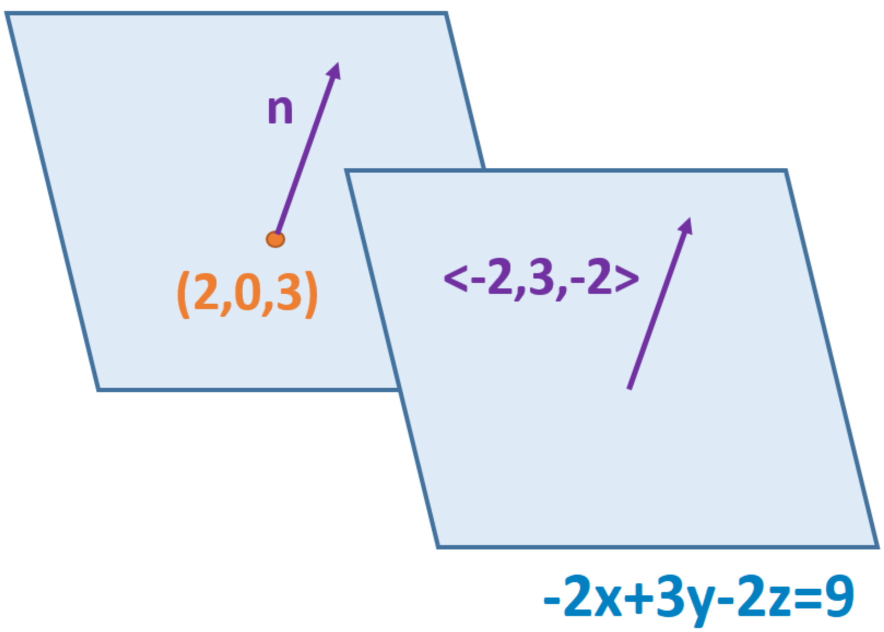
So $\mathbf{n} = \langle 9, -21, 48 \rangle$

(3) **Equation:** (Recall the point was $A = (2, 1, 2)$)

$$9(x - 2) - 21(y - 1) + 48(z - 2) = 0$$

Example 3:

Find the equation of the plane parallel to $-2x + 3y - 2z = 9$ which passes through the point $(2, 0, 3)$.



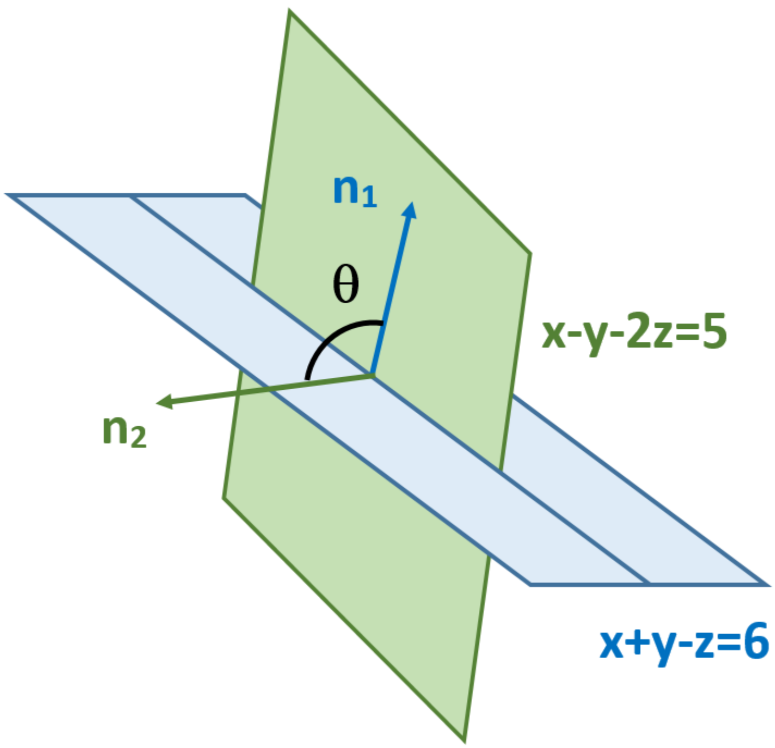
(1) **Point:** $(2, 0, 3)$

(2) **Normal Vector:** Two parallel planes have the same normal vector, hence $\mathbf{n} = \langle -2, 3, -2 \rangle$

(3) **Equation:** $-2(x - 2) + 3(y - 0) - 2(z - 3) = 0$

Example 4:

Find the angle between the planes $x+y-z=6$ and $x-y-2z=5$



The normal vector to the plane $x+y-z=6$ is $\mathbf{n_1} = \langle 1, 1, -1 \rangle$ and the normal vector to the plane $x-y-2z=5$ is $\mathbf{n_2} = \langle 1, -1, -2 \rangle$

The angle between the two planes is nothing other than the angle between the normal vectors. Therefore, by the angle formula, we have

$$\mathbf{n_1} \cdot \mathbf{n_2} = \|\mathbf{n_1}\| \, \|\mathbf{n_2}\| \cos(\theta)$$

However, we have:

$$\begin{aligned}\mathbf{n_1} \cdot \mathbf{n_2} =& 1 - 1 + 2 = 2\\ \|\mathbf{n_1}\| =& \sqrt{1+1+1} = \sqrt{3}\\ \|\mathbf{n_2}\| =& \sqrt{1+1+4} = \sqrt{6}\end{aligned}$$

Therefore we get:

$$\begin{aligned}2 =& \sqrt{3}\sqrt{6}\cos(\theta)\\ \cos(\theta) =& \frac{2}{\sqrt{18}}\\ \theta =& \cos^{-1}\left(\frac{2}{\sqrt{18}}\right)\\ \theta \approx& 62^\circ\end{aligned}$$

Note: You can find more examples and practice problems below, in case we don't get to them during lecture.

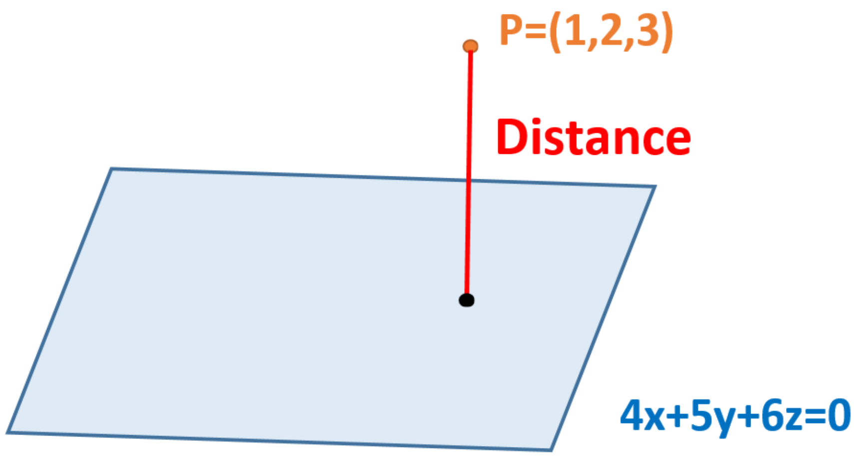
3. Distance between point and plane

Video: Distance between point and plane

Finally, let's talk about the celebrated distance formula, which gives a nice formula for the distance between a point and a plane.

Example 5:

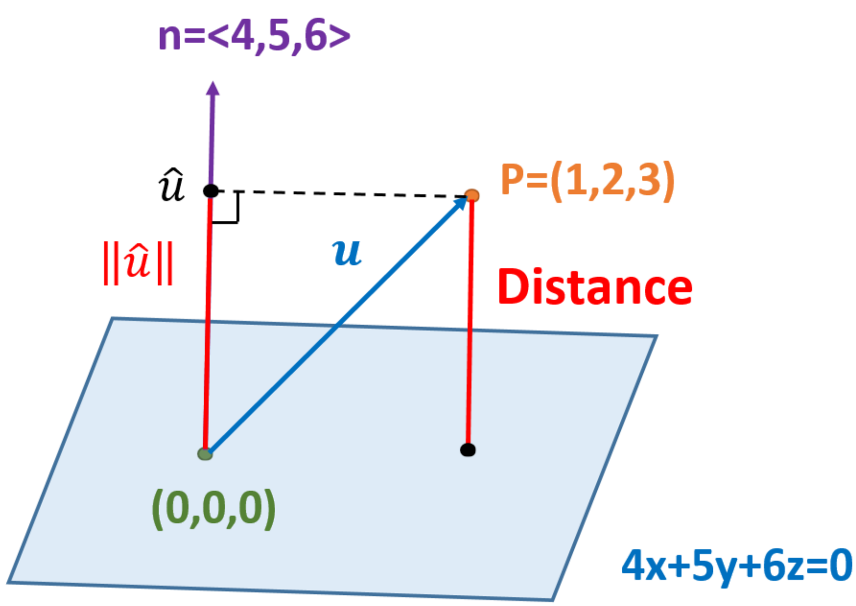
Find the distance between the point $P = (1, 2, 3)$ and the plane $4x + 5y + 6z = 0$



(Notice here that $(0,0,0)$ is in the plane)

Trick: Let $\mathbf{u} = \langle 1,2,3 \rangle$ (vector from $(0,0,0)$ to P) and consider:

$$\hat{u} = \text{proj}_{\mathbf{n}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n}$$



Then, according to the picture above, the distance is given precisely by $\|\hat{\mathbf{u}}\|$, and so:

$$
\begin{aligned}
\text{Distance} &= \|\hat{\mathbf{u}}\| \\
&\overset{\text{DEF}}{=} |\text{comp}_{\mathbf{n}}\,\mathbf{u}| \quad \text{(scalar projection)} \\
&\overset{\text{DEF}}{=} \frac{|\mathbf{n}\cdot\mathbf{u}|}{\|\mathbf{n}\|} \\
&= \frac{|\langle 4,5,6\rangle \cdot \langle 1,2,3\rangle|}{\sqrt{4^2+5^2+6^2}} \\
&= \frac{4(1)+5(2)+6(3)}{\sqrt{4^2+5^2+6^2}} \\
&= \frac{32}{\sqrt{77}}
\end{aligned}
$$

More generally, we obtain the celebrated distance formula (see this video for an optional full proof):

Distance Formula:

The distance between the point $P = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$
\text{Distance} = \frac{|a(x_0) + b(y_0) + c(z_0) + d|}{\sqrt{a^2+b^2+c^2}}
$$

Example 6:

Find the distance between the point $P = (2, 3, -1)$ and the plane $3x + 2y - 3z = 2$

The equation of the plane can be written as $3x + 2y - 3z - 2 = 0$ and $P = (2, 3, -1)$ and hence

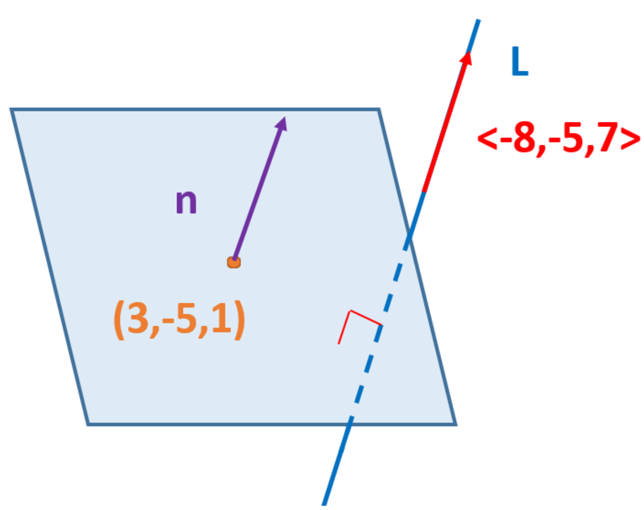
$$\begin{aligned}\text{Distance } &= \frac{|(3)(2)+(2)(3)+(-3)(-1)+(-2)|}{\sqrt{3^2+2^2+(-3)^2}}\\ &= \frac{|6+6+3-2|}{\sqrt{9+4+9}}\\ &= \frac{13}{\sqrt{22}}\end{aligned}$$

4. More Practice

Note: There are more examples here than can comfortably fit in one lecture, but they are all **important** for your homework and quizzes/exams, so make sure to look at all of them.

Example 7:

Find the equation of the plane passing through $(3,-5,1)$ and perpendicular to the line L with equation $\mathbf{r}(t) = \langle 2-8t, 3-5t, 6+7t\rangle$



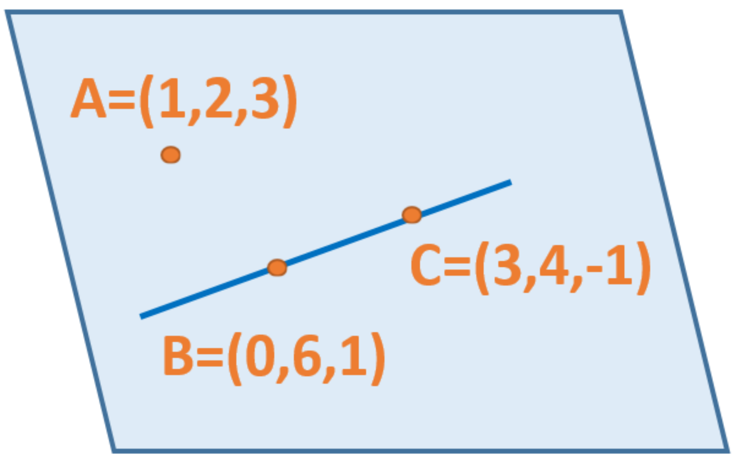
(1) **Point:** $(3, -5, 1)$

(2) **Normal Vector:** By the picture above, the *normal* vector to the plane is the *same* as the *direction* vector of the line, which is $\langle -8, -5, 7\rangle$, hence $\mathbf{n} = \langle -8, -5, 7\rangle$

(3) **Equation:** $-8(x-3) - 5(y+5) + 7(z-1) = 0$

Example 8:

Find an equation of the plane through $(1, 2, 3)$ and containing the line represented with equation $\mathbf{r}(t) = \langle 3t, 6-2t, 1-2t\rangle$



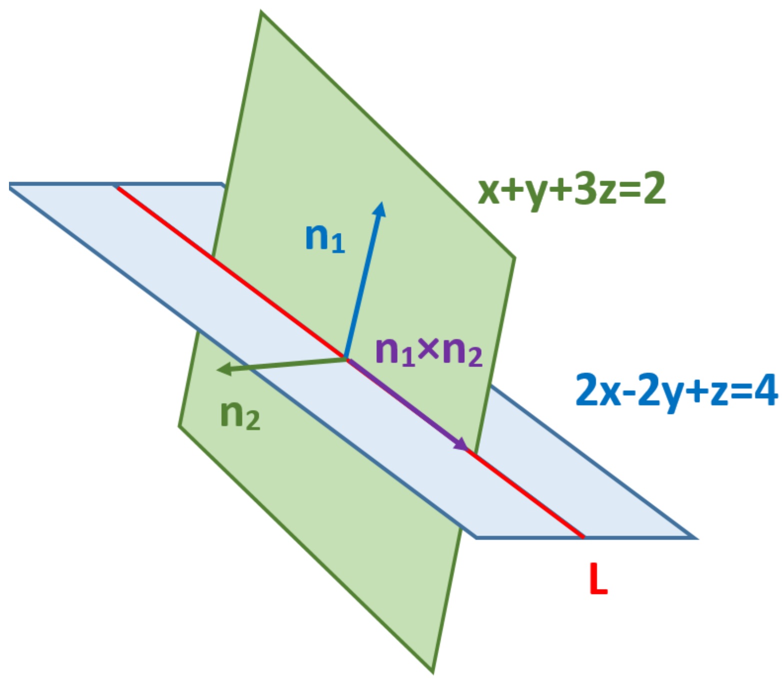
Outline: This is simpler than it sounds! All you really need to do is find 3 points on the plane.

One point is $(1, 2, 3)$, and for the other two points, just find two other points on the line, for example $\mathbf{r}(0) = \langle 0, 6, 1\rangle$ and $\mathbf{r}(1) = \langle 3(1), 6-2, 1-2\rangle = \langle 3, 4, -1\rangle$ (any other value works just fine)

Then find the equation of the plane that contains $A = (1,2,3)$, $B = (0,6,1)$ and $C = (3,4,-1)$, which you can do as in the example above (calculate $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and take cross products etc.)

Example 9: (Good Quiz/Exam Question)

Find the vector equation of the line of intersection of the planes $2x - 2y + z = 4$ and $x + y + 3z = 2$



(1) **Point:** For this we need *one* solution of

$$\begin{cases} 2x - 2y + z = 4 \\ x + y + 3z = 2 \end{cases}$$

For example, if we let $z = 0$, we get

$$\begin{cases} 2x - 2y + 0 = 4 \\ x + y + 0 = 2 \end{cases} \Rightarrow \begin{cases} x - y + 0 = 2 \\ x + y + 0 = 2 \end{cases} \Rightarrow \begin{cases} 2x = 4 \\ -2y = 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 0 \end{cases}$$

Which gives the point $(2, 0, 0)$ (since $z = 0$)

(2) **Direction Vector:** The normal vector to the first plane is $\mathbf{n_1} = \langle 2, -2, 1 \rangle$ and the normal vector to the second plane is $\mathbf{n_2} = \langle 1, 1, 3 \rangle$.

By the picture above, the direction vector to the line is perpendicular to both $\mathbf{n_1}$ and $\mathbf{n_2}$, hence given by

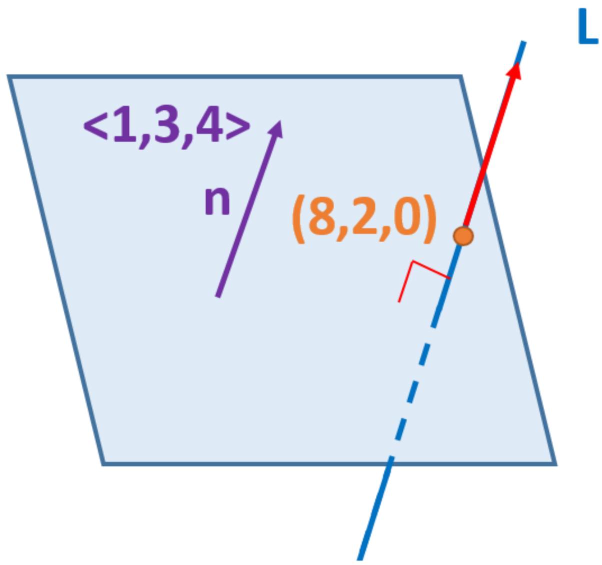
$$\mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ 1 & 1 & 3 \end{vmatrix} = (-6-1)\mathbf{i} - (6-1)\mathbf{j} + (2+2)\mathbf{k} = -7\mathbf{i} - 5\mathbf{j} + 4\mathbf{k} = \langle -7, -5, 4 \rangle$$

(3) **Equation:**

$$\mathbf{r(t)} = \langle 2 - 7t, -5t, 4t \rangle$$

Example 10:

Find an equation of the line through $(8, 2, 0)$ and perpendicular to the plane $x + 3y + 4z = 5$.



Not *quite* the same as before, since need to find the equation of a *line*

(1) **Point:** $(8, 2, 0)$

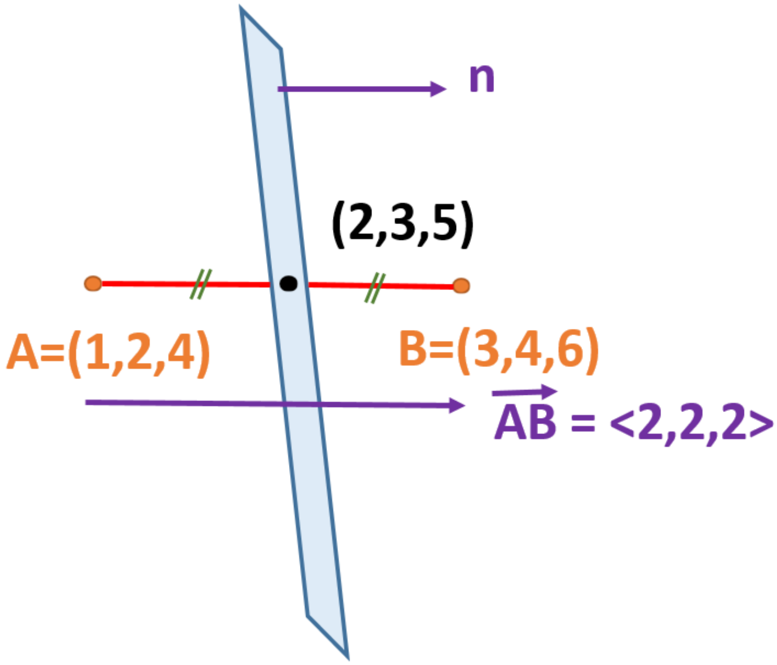
(2) **Direction Vector:** By the picture above, the direction vector of the line is the same as the normal vector of the plane, which is $\langle 1, 3, 4 \rangle$

(3) **Equation:**

$$\begin{cases} x(t) = 8 + t \\ y(t) = 2 + 3t \\ z(t) = 4t \end{cases}$$

Example 11:

Find an equation of the plane consisting of all points that are equidistant (= the same distance) to $A = (1, 2, 4)$ and $B = (3, 4, 6)$



(1) **Point:** Notice that the midpoint of A and B is on the plane (since it's the same distance from A to B)

$$\text{Midpoint } = \left(\frac{1+3}{2}, \frac{2+4}{2}, \frac{4+6}{2}\right) = (2, 3, 5)$$

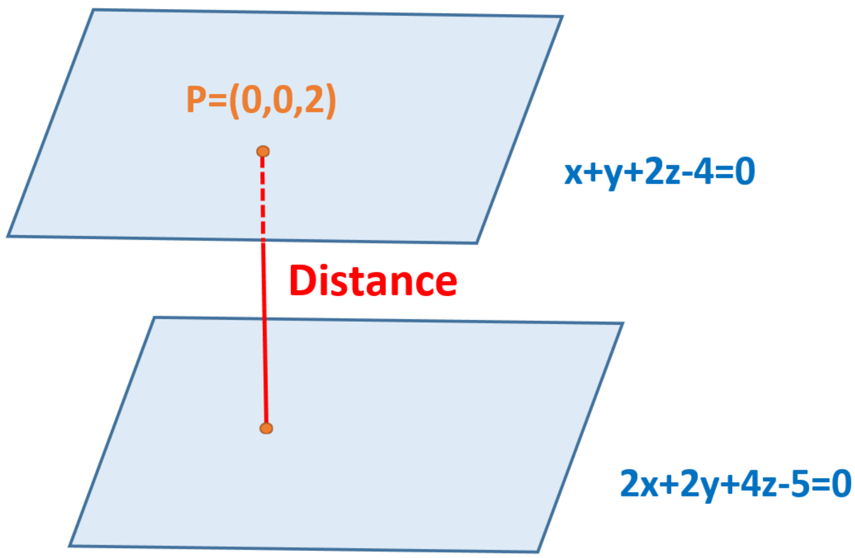
(2) **Normal Vector:** By the picture, the normal vector is (parallel) to $\overrightarrow{AB} = \langle 3-1, 4-2, 6-4\rangle = \langle 2, 2, 2\rangle$, so $\mathbf{n} = \langle 2, 2, 2\rangle$ ($\mathbf{n} = \langle 1, 1, 1\rangle$ is ok too)

(3) **Equation:**

$$\begin{aligned} 2(x-2) + 2(y-3) + 2(z-5) &= 0 \\ (x-2) + (y-3) + (z-5) &= 0 \\ x + y + z &= 10 \end{aligned}$$

Example 12:

Find the distance between the (parallel) planes $x+y+2z-4=0$ and $2x+2y+4z-5=0$



Not as bad as it sounds!

STEP 1: Find *one* point on *one* plane.

For example, take $x+y+2z-4=0$. Then for example set $x=0$ and $y=0$, and you get $0+0+2z-4=0$ so $z=2$, which gives $P=(0,0,2)$

STEP 2: The distance is simply the distance between $P=(0,0,2)$ and the other plane $2x+2y+4z-5=0$, which is:

$$\text{Distance} = \frac{|(2)(0)+2(0)+4(2)-5|}{\sqrt{2^2+2^2+4^2}} = \frac{|8-5|}{\sqrt{4+4+16}} = \frac{3}{\sqrt{24}}$$