LECTURE 6: LIMITS THEOREMS FOR SEQUENCES

Let's now show some properties of limits, starting with:

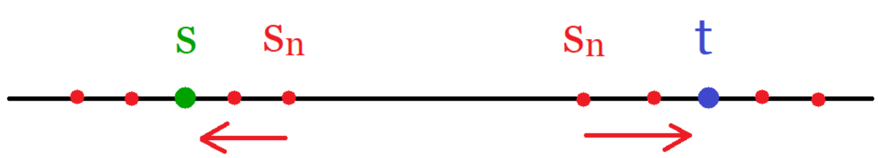
1. Limits are Unique (Section 7)

Video: Limits are Unique

Limits Are Unique:

If $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_n = t$, then $s = t$

In other words, s_n cannot converge to two different limits at the same time. In a way this makes sense: How can s_n be close to both s and t?



Proof: Suppose $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_n = t$.

Let $\epsilon > 0$ be arbitrary.

Since $\lim_{n\to\infty} s_n = s$, we know that there is $N_1 > 0$ such that if $n > N_1$, then $|s_n - s| < \epsilon$

Date: Thursday, September 16, 2021.

Since $\lim_{n\to\infty} s_n = t$, we know that there is $N_2 > 0$ such that if $n > N_2$, then $|s_n - t| < \epsilon$

Let $N = \max\{N_1, N_2\}$, then if $n > N$, we get:

$$|s - t| = |(s - s_n) + (s_n - t)| \leq |s - s_n| + |s_n - t| < \epsilon + \epsilon = 2\epsilon$$

Therefore $0 \leq |s - t| < 2\epsilon$, but since ϵ is arbitrarily small, we get $|s - t| = 0 \Rightarrow s = t$ □

2. Convergent Sequences are Bounded

Video: Convergent Sequences are Bounded

Here's another quick but neat fact about convergent sequences:

Convergent sequences are bounded:

If s_n converges (to s), then s_n is bounded, that is there is M such that

$$|s_n| \leq M \quad \text{for all } n \in \mathbb{N}$$

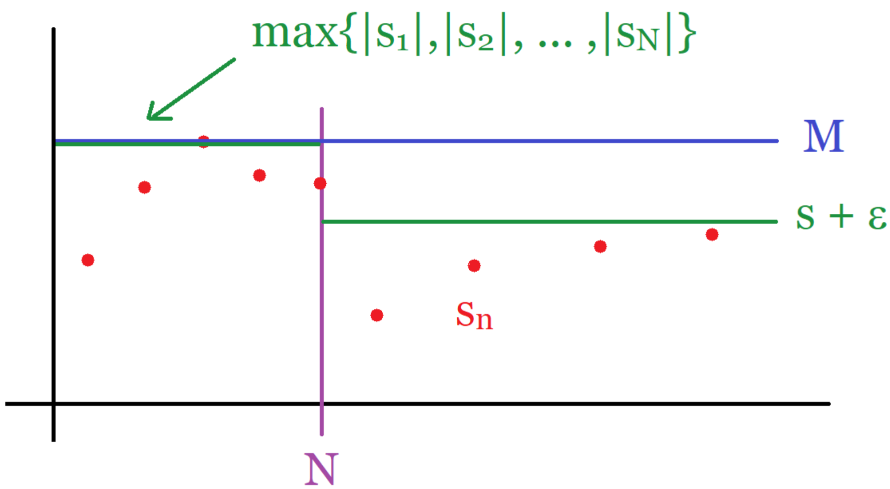
Proof: Fix $\epsilon > 0$, then since $s_n \to s$, there is N (WLOG assume $N \in \mathbb{N}$) such that if $n > N$, then $|s_n - s| < \epsilon$.

But if $n > N$, then

$$|s_n| = |s_n - s + s| \leq |s_n - s| + |s| < \epsilon + |s|$$

Note: Intuitively, for $n > N$, s_n is bounded by the fixed number $\epsilon + |s|$. And for $n \leq N$, s_n is just bounded above by the largest one of

$s_1, s_2, \cdots, s_N$ (which are just finitely many terms).



Let $M = \max\{|s_1|, \cdots, |s_N|, \epsilon + |s|\}$ and let's show $|s_n| \leq M$ for all n

Case 1: $n > N$, then $|s_n| \leq \epsilon + |s| \leq M$.

Case 2: $n \leq N$, then $|s_n| \leq \max\{|s_1|, \cdots, |s_N|\} \leq M$.

So in any case, $|s_n| \leq M$ ✓ □

3. Sum of Limits

Video: Sum of Limits

Finally, since we're more comfortable with the definition of the limit, we can prove some limit laws, starting with:

Sum of Limits:

Suppose $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, then

$$\lim_{n\to\infty} s_n + t_n = s + t$$

In other words:

$$\lim_{n\to\infty} s_n + t_n = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$$

Proof: Let $\epsilon > 0$ be given

Since $\lim_{n\to\infty} s_n = s$, we know that there is N_1 such that if $n > N_1$, then $|s_n - s| < \frac{\epsilon}{2}$.

And since $\lim_{n\to\infty} t_n = t$, there is N_2 such that if $n > N_2$, then $|t_n - t| < \frac{\epsilon}{2}$.

But if $N = \max\{N_1, N_2\}$ then if $n > N$, we have:

$$|s_n + t_n - (s+t)| = |s_n - s + t_n - t| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \checkmark$$

Hence $\lim_{n\to\infty} s_n + t_n = s + t$ $\square$

4. Product of Limits

Video: Product of Limits

In the same spirit, let's show that products of sequences converge:

Product of Limits:

Suppose $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, then

$$\lim_{n\to\infty} s_n t_n = st = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right)$$

Proof:

STEP 1: Scratch-work

This relies on a clever but *important* use of the triangle inequality:

$$\begin{aligned} |s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\ &\leq |s_n t_n - s_n t| + |s_n t - st| \\ &= |s_n|\,|t_n - t| + |t|\,|s_n - s| \end{aligned}$$

Now since $|t_n - t|$ is small and (s_n) is bounded, the first term is small, and since $|s_n - s|$ is small, the second term is small as well.

STEP 2: Assume $t \neq 0$ (the case $t = 0$ is dealt with separately, see Problem 4 in section 8)

Let $\epsilon > 0$ be given

Since (s_n) converges, (s_n) is bounded, so there is $M > 0$ such that $|s_n| \leq M$ for all n.

Since $t_n \to t$, there is N_1 such that if $n > N_1$, then $|t_n - t| < \frac{\epsilon}{2M}$ (This will eventually cancel the $|s_n|$ term)

Since $s_n \to s$, there is N_2 such that is $n > N_2$, then $|s_n - s| < \frac{\epsilon}{2|t|}$ (This will eventually cancel the $|t|$ term)

Let $N = \max\{N_1, N_2\}$, then if $n > N$, we have

$$\begin{aligned} |s_n t_n - st| &\leq |s_n| \, |t_n - t| + |t| \, |s_n - s| \\ &\leq M \, |t_n - t| + |t| \, |s_n - s| \\ &< M \left(\frac{\epsilon}{2M} \right) + |t| \left(\frac{\epsilon}{2\,|t|} \right) \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

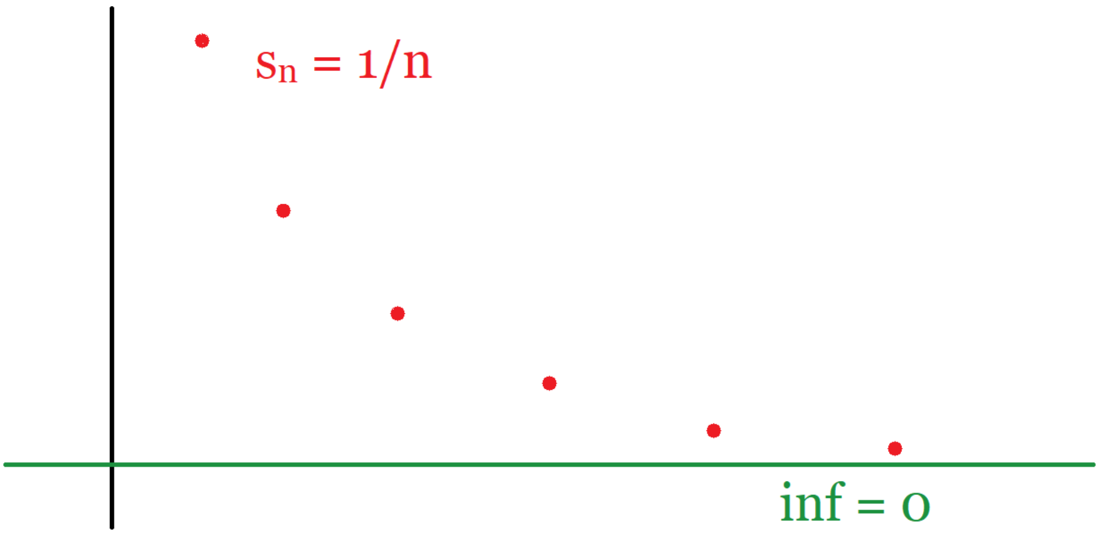
Therefore $\lim_{n\to\infty} s_n t_n = st$ □

5. Bounded Away from 0

Video: Bounded Away from 0

Before we prove the analogous result for quotients, we need another fact about sequences:

Motivation: Consider $s_n = \frac{1}{n}$



Even though every term is positive, we do **NOT** have $\inf \{s_n \mid n \in \mathbb{N}\} > 0$. In fact, here the inf is 0.

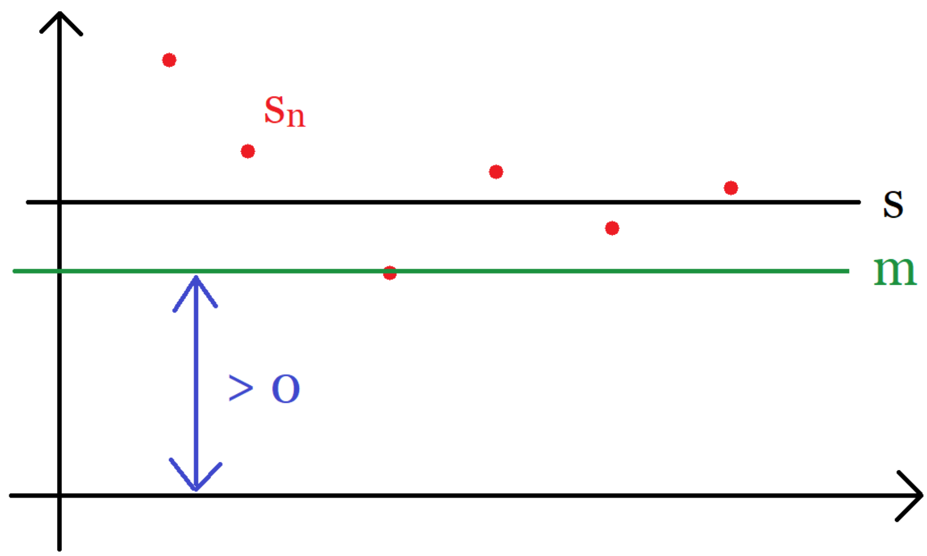
This is not the case with sequences $s_n \to s$ with $s \neq 0$:

Fact:

Suppose $s_n \neq 0$ for all n and $\lim_{n\to\infty} s_n = s$ with $s \neq 0$. Then there is some $m > 0$ such that

$$|s_n| \geq m \qquad \text{(For all } n\text{)}$$

In particular, $\inf \{|s_n| \mid n \in \mathbb{N}\} \geq m > 0$ and so $|s_n|$ is "bounded away from 0"



Note: This is **FALSE** if $s = 0$, as the example $s_n = \frac{1}{n}$ shows.

Proof: WLOG[1], assume $s > 0$

Let $\epsilon > 0$ TBA

Since $s_n \to s$, there is N (Assume $N \in \mathbb{N}$)[2] such that if $n > N$ then $|s_n - s| < \epsilon$. However,

$$|s_n - s| < \epsilon \Rightarrow -\epsilon < s_n - s < \epsilon \Rightarrow s - \epsilon < s_n < s + \epsilon$$

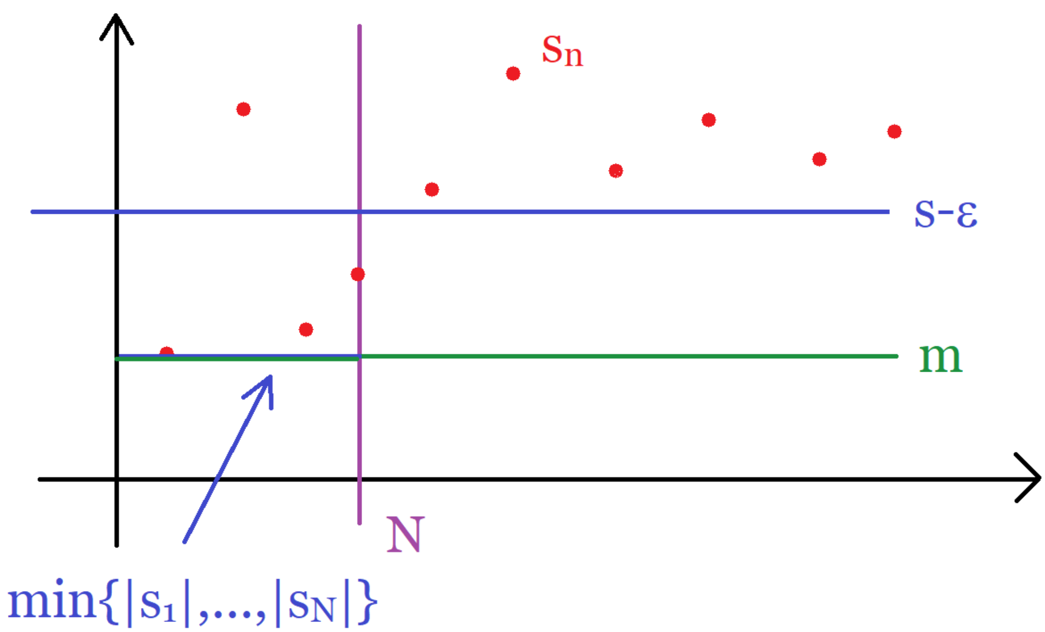
In particular $s_n > s - \epsilon$.

Now *choose* $\epsilon > 0$ such that $s - \epsilon > 0$, that is $\epsilon < s$ (which we can do since $s > 0$)

[1] If $s < 0$, just replace s_n with $-s_n \neq 0$ which doesn't change the result since $|-s_n| = |s_n|$)

[2] If N is not an integer, just replace N by the smallest integer greater than N. So if $N = 3.14$, replace N with 4

Intuitively: For $n > N$, s_n is bounded below by the fixed number $s - \epsilon > 0$. And for $n \leq N$, $|s_n|$ is just bounded below by the smallest one of $|s_1|, |s_2|, \cdots, |s_N|$ (which are just finitely many terms).



Let $m = \min\{|s_1|, \cdots, |s_N|, s - \epsilon\} > 0$ and let's show that $|s_n| \geq m$ for all n.

Case 1: If $n > N$, then $s_n > s - \epsilon \geq m (> 0) \Rightarrow |s_n| \geq m$

Case 2: If $n \leq N$, then

$$|s_n| \geq \min\{|s_1|, \cdots, |s_N|\} \geq \min\{|s_1|, \cdots, |s_N|, s - \epsilon\} = m$$

Therefore $|s_n| \geq m$ for all n ✓

6. Quotient of Limits (Example 7)

Video: Limit Example 7: Quotients

Now we would like to prove that $\lim_{n\to\infty} \frac{t_n}{s_n} = \frac{t}{s}$, and for this we need an important special case:

Example 7:

Show that if $s_n \neq 0$ for all n and $\lim_{n\to\infty} s_n = s \neq 0$, then

$$\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$$

(Example 6 is the example with $|s_n| \to |s|$, AP on the section 8 HW)

STEP 1: Scratch work

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{(s_n)s}\right| = \frac{|s_n - s|}{|s_n|\,|s|} < \epsilon$$

If the denominator were independent of n, we could just solve for $|s_n - s|$. But remember that (s_n) is bounded away from 0, that is $|s_n| \geq m$ for some $m > 0$, so we get

$$\frac{|s_n - s|}{|s_n|\,|s|} < \frac{|s_n - s|}{m\,|s|} < \epsilon$$

Which gives $|s_n - s| < m\,|s|\,\epsilon$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given.

Since (s_n) is bounded away from 0, there is $m > 0$ such that $|s_n| \geq m$ for all n.

Now since $s_n \to s$, there is n such that for all n, if $n > N$, then

$$|s_n - s| < m\,|s|\,\epsilon$$

But then, with the same n, if $n > N$, we get:

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s_n - s|}{|s_n|\,|s|} < \frac{|s_n - s|}{m\,|s|} < \frac{\epsilon m\,|s|}{m\,|s|} = \epsilon\checkmark$$

Hence $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ □

Corollary:

If $s_n \neq 0$ for all n and $s_n \to s \neq 0$ and $t_n \to t$, then

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \frac{t}{s}$$

In other words:

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} s_n}$$

Proof:

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \lim_{n\to\infty} t_n \left(\frac{1}{s_n}\right) = \left(\lim_{n\to\infty} t_n\right)\left(\lim_{n\to\infty} \frac{1}{s_n}\right) = t\left(\frac{1}{s}\right) = \frac{t}{s}$$

Where, we used both the product law and Example 7 above. □

From now on, feel free to manipulate limits the same way you do in calculus!

7. Example 8: Exponential Limit

Video: Limit Example 8: Exponential

Now let's calculate some classical limits. For the following, we'll need the binomial theorem:

Motivation: $(a+b)^2 = a^2 + 2ab + b^2$, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, and more generally:

Binomial Theorem:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + b^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$$

Where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Example:

$$\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad (p > 0)$$

(SKIP, similar to Exercise 1(b) in section 8)

Example 8:

$$\text{If } |a| < 1, \text{ then } \lim_{n\to\infty} a^n = 0$$

For example, this shows $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$