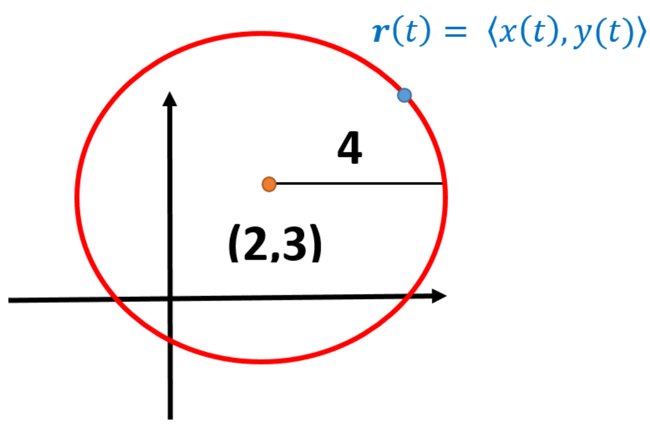
LECTURE 6: VECTOR FUNCTIONS (I)

Welcome to the world of vector functions, which are just parametric equations in disguise.

1. Definition and Examples

Example 1:

Find the **vector equations** of the circle centered at $(2, 3)$ and radius 4.



Parametric Equations:

$$\begin{cases} x(t) = 2 + 4\cos(t) \\ y(t) = 3 + 4\sin(t) \\ \qquad 0 \leq t \leq 2\pi \end{cases}$$

Date: Friday, September 10, 2021.

Vector Equation: Just put $x(t)$ and $y(t)$ in a vector:

Definition: (Vector Function)

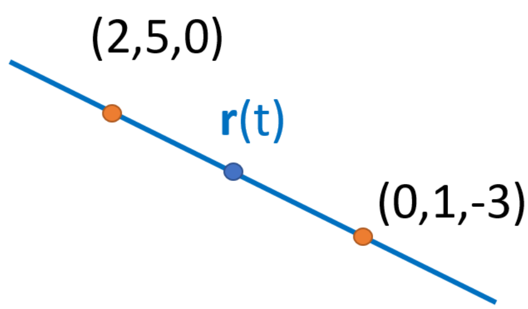
$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle 2 + 4\cos(t), 3 + 4\sin(t) \rangle$$

Here for every time t, we have a *vector* $\mathbf{r}(t)$

If the notation $\mathbf{r}(t)$ looks familiar, it's because we used it for lines:

Example 2:

Find the vector equation of the line going through $(2, 5, 0)$ and $(0, 1, -3)$



Point: $(2, 5, 0)$

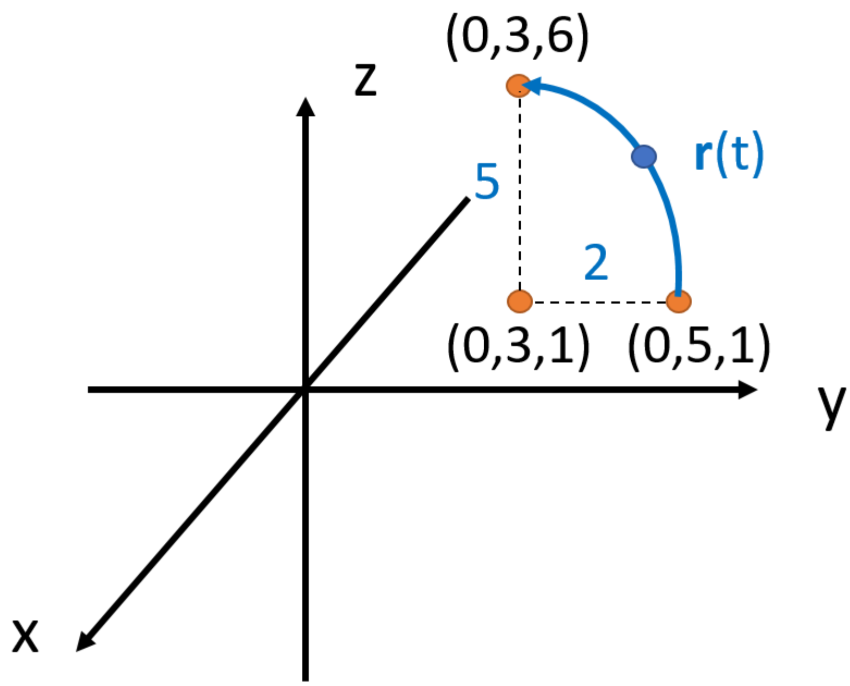
Direction Vector: $\langle 0 - 2, 1 - 5, -3 - 0 \rangle = \langle -2, -4, -3 \rangle$

Vector Equation:

$$\mathbf{r}(t) = \langle 2, 5, 0 \rangle + t \langle -2, -4, -3 \rangle = \langle 2 - 2t, 5 - 4t, -3t \rangle$$

Example 3: (extra practice)

Find the vector equation of the quarter-ellipse centered at $(0,3,1)$ and going from $(0,5,1)$ to $(0,3,6)$ (counterclockwise)



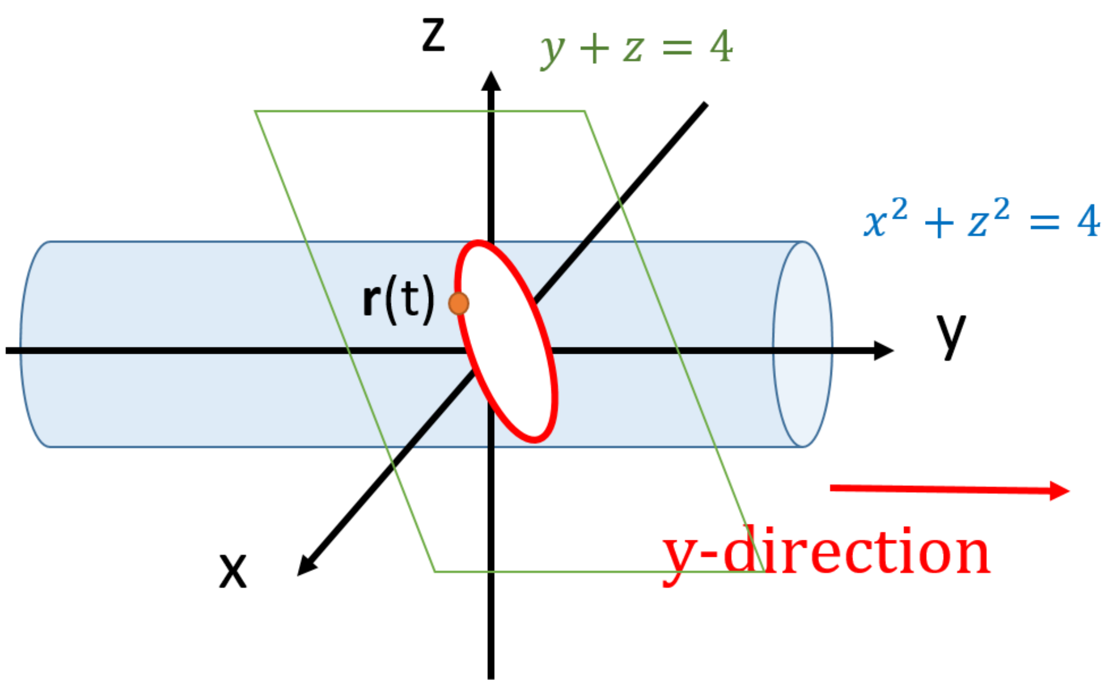
$$\begin{cases} x(t) = 0 \\ y(t) = 3 + 2\cos(t) \\ z(t) = 1 + 5\sin(t) \\ \quad 0 \leq t \leq \dfrac{\pi}{2} \end{cases}$$

$$\mathbf{r}(t) = \langle 0, 3 + 2\cos(t), 1 + 5\sin(t) \rangle, 0 \leq t \leq \frac{\pi}{2}$$

Example 4:

Find the vector equation of the curve of intersection of the cylinder $x^2 + z^2 = 4$ and the plane $y + z = 4$

(Notice that in $x^2 + z^2 = 4$, y is missing, so it's a cylinder in the $y-$direction. And for $y + z = 4 \Rightarrow z = 4 - y$, draw the line $z = 4 - y$ and shift it in the $x-$direction)



First, notice that in the $xz-$plane, $x^2 + z^2 = 4$ is just a circle of radius 2 centered at $(0, 0, 0)$, which gives

$$x(t) = 2\cos(t)$$
$$z(t) = 2\sin(t)$$

To figure out y, simply use

$$y + z = 4 \Rightarrow y = 4 - z \Rightarrow y(t) = 4 - z(t) = 4 - 2\sin(t)$$

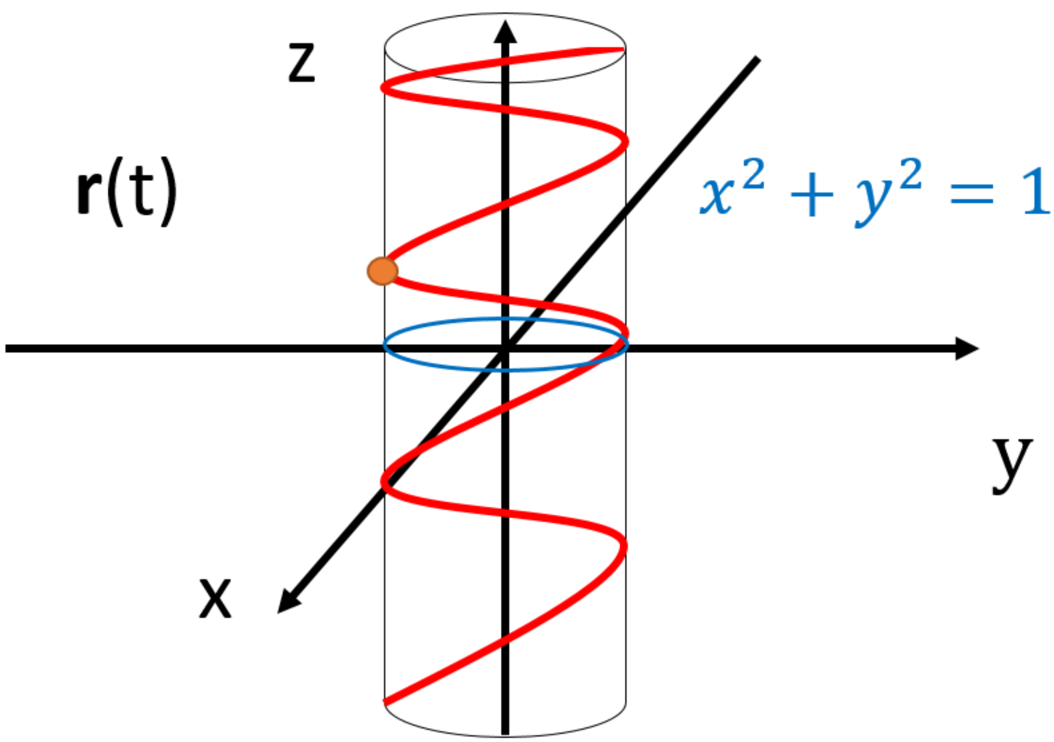
$$\mathbf{r}(t) = \langle 2\cos(t), 4 - 2\sin(t), 2\sin(t)\rangle \ (0 \leq t \leq 2\pi)$$

Finally, it's often useful to be able to sketch some vector curves.

Example 5:

Sketch the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), t\rangle$

Notice here that $x(t) = \cos(t)$ and $y(t) = \sin(t)$, so $x^2 + y^2 = 1$, which means that our curve lies in the *cylinder* $x^2 + y^2 = 1$. Finally $z(t) = t$ just means t is going up (and down), so the curve is a helix/DNA/slinky:



Slinky

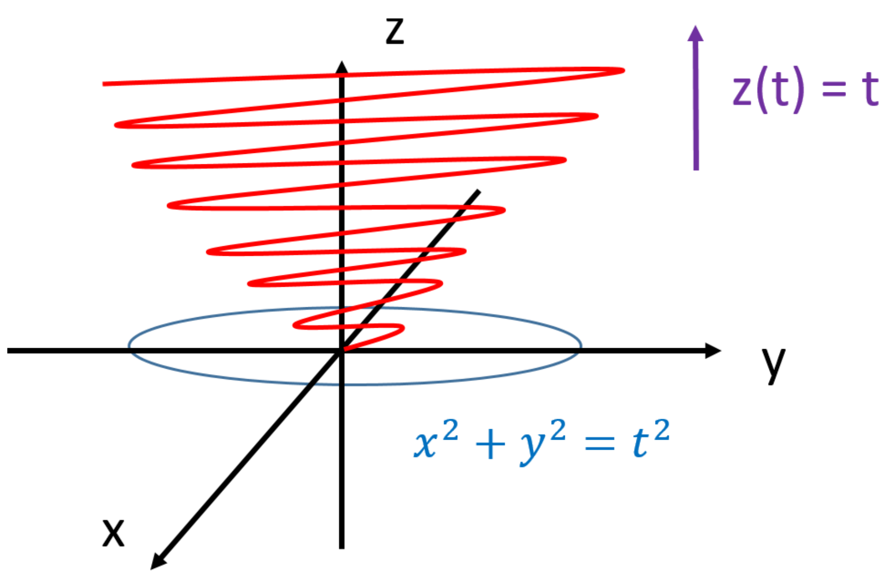
Example 6: (extra practice)

Sketch the curve $\mathbf{r}(t) = \langle t\cos(t), t\sin(t), t \rangle, t \geq 0$

Here notice that:

$$x^2 + y^2 = t^2\cos^2(t) + t^2\sin^2(t) = t^2$$

So x and y lie on a circle with radius t (which gets bigger and bigger), and z is always increasing, so the curve is a tornado in the $z-$direction:



2. Calculus with Vector Functions

What can we do with vector functions? The good news is that *all* the concepts from calculus (limits, derivatives, integrals) easily apply to vector functions as well!

2.1. Limits.

We can take limits of vector functions:

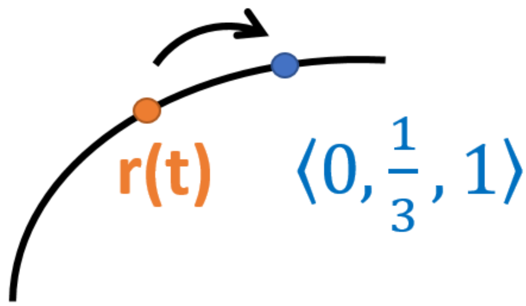
Example 7:

Find $\lim_{t\to 0}\mathbf{r}(t)$ where

$$\mathbf{r}(t) = \left\langle \ln(1+t), \frac{1}{\sqrt{9-t^2}}, 2^t \right\rangle$$

$$\begin{aligned}
\lim_{t\to 0}\mathbf{r}(t) &= \lim_{t\to 0}\left\langle \ln(1+t), \frac{1}{\sqrt{9-t^2}}, 2^t \right\rangle \\
&\overset{\text{DEF}}{=} \left\langle \lim_{t\to 0}\ln(1+t), \lim_{t\to 0}\frac{1}{\sqrt{9-t^2}}, \lim_{t\to 0}2^t \right\rangle \\
&= \left\langle \ln(1+0), \frac{1}{\sqrt{9-0^2}}, 2^0 \right\rangle \\
&= \left\langle 0, \frac{1}{3}, 1 \right\rangle
\end{aligned}$$

Interpretation: As t goes to 0, $\mathbf{r}(t)$ gets closer to $\left\langle 0, \frac{1}{3}, 1 \right\rangle$



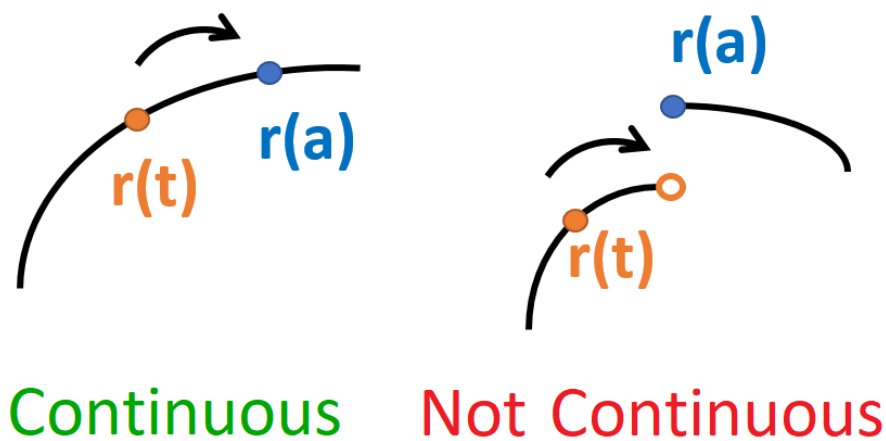
Example 8: (optional)

Is $\mathbf{r}(t)$ (as above) **continuous** at $t = 0$?

Definition:

$\mathbf{r}(t)$ is **continuous** at a if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$



Here $a = 0$ and $\lim_{t \to 0} \mathbf{r}(t) = \left\langle 0, \frac{1}{3}, 1 \right\rangle$ (by the previous example)

Moreover:

$$\mathbf{r}(0) = \left\langle \ln(1+0), \frac{1}{\sqrt{9-0^2}}, 2^0 \right\rangle = \left\langle 0, \frac{1}{3}, 1 \right\rangle$$

Since both of those are equal, the answer is **YES**.

2.2. **Integrals.** We can take also integrals of vector functions:

Example 9:

Find $\int \mathbf{r}(t)dt$ where $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$

$$\begin{aligned}\int \mathbf{r}(t)dt &\overset{\text{DEF}}{=} \left\langle \int t dt, \int t^2 dt, \int t^3 dt \right\rangle \\ &= \left\langle \frac{t^2}{2} + A, \frac{t^3}{3} + B, \frac{t^4}{4} + C \right\rangle\end{aligned}$$

Note: Make sure **NOT** to write $\left\langle \frac{t^2}{2} + C, \frac{t^3}{3} + C, \frac{t^4}{4} + C \right\rangle$. The constants *could* in theory be different!

(This unfortunately does not measure the area under the curve because dt is a small change in time, not a small change in x; we'll later learn how to do that.)

2.3. **Derivatives.** Most importantly, we can take derivatives of vector functions, which will have an important interpretation.

Example 10:

Find $\mathbf{r}'(t)$ where $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$

$$\mathbf{r}'(t) \overset{\text{DEF}}{=} \left\langle (t)', \left(t^2\right)', \left(t^3\right)' \right\rangle = \langle 1, 2t, 3t^2 \rangle$$

Example 11:

Find $\mathbf{r}'(\pi)$ where $\mathbf{r}(t) = \langle \cos(t), \sin(t), \sin(2t) \rangle$

$$\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 2\cos(2t) \rangle$$

$$\mathbf{r}'(\pi) = \langle -\sin(\pi), \cos(\pi), 2\cos(2\pi) \rangle = \langle 0, -1, 2 \rangle$$

Example 12: (extra practice)

Find $\mathbf{r}''(t)$ where $\mathbf{r}(t) = \left\langle e^t, e^{2t}, e^{4t} \right\rangle$

$$\mathbf{r}'(t) = \left\langle e^t, 2e^{2t}, 4e^{4t} \right\rangle$$

$$\mathbf{r}''(t) = \left\langle e^t, 2\left(2e^{2t}\right), 4\left(4e^{4t}\right) \right\rangle = \left\langle e^t, 4e^{2t}, 16e^{4t} \right\rangle$$

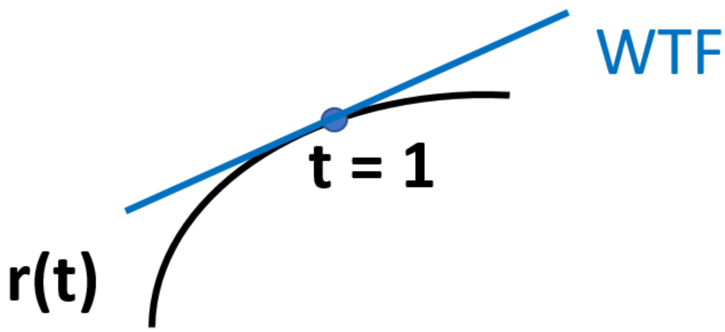
3. Tangent Lines

Why are derivatives so important? Because they help us find tangent lines to curves.

Example 13: (Good quiz/exam question)

Find the (parametric) equations of the tangent line to the following curve at $t = 1$:

$$\mathbf{r}(t) = \left\langle 1 + t, t^2, 3t + t^3 \right\rangle$$



Recall:

To find the equation of a line, we need a **point** and a **direction vector**.

Point: Since $t = 1$, the point is

$$\mathbf{r}(1) = \left\langle 1+1, 1^2, 3(1)+1^3 \right\rangle = \langle 2, 1, 4 \rangle$$

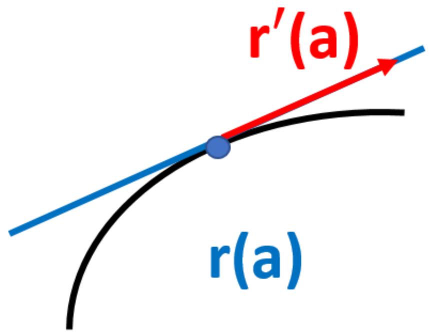
Direction Vector: First calculate

$$\mathbf{r}'(t) = \left\langle 1, 2t, 3+3t^2 \right\rangle$$

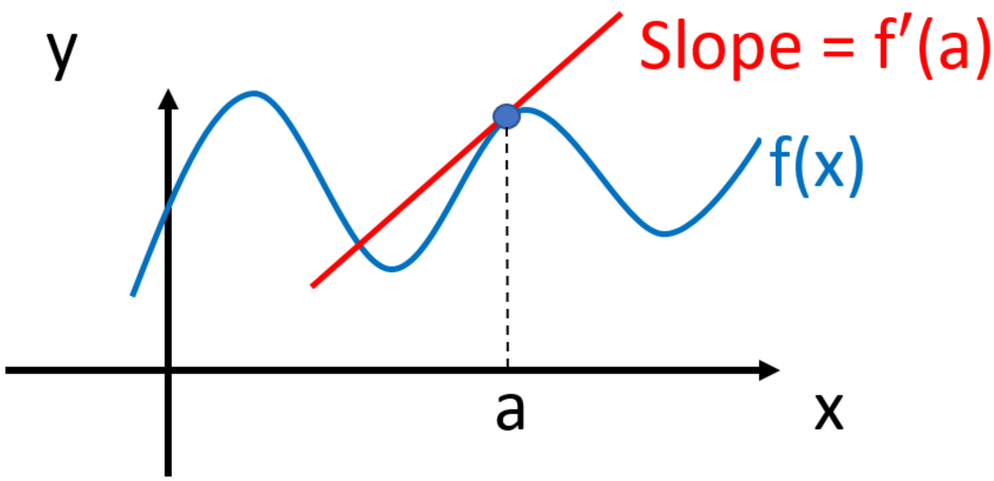
Definition:

$\mathbf{r}'(a)$ is the **tangent vector** of $\mathbf{r}$ at $t = a$

Interpretation: $\mathbf{r}'(a)$ gives the **direction/slope** vector of the tangent line to the curve $\mathbf{r}(t)$ at $t = a$:



Compare this to single-variable calculus, where $f'(a)$ gives the slope of the tangent line to a function f at a point a.



Here: $t = 1$ so

$$\mathbf{r}'(1) = \left\langle 1, 2(1), 3 + 3(1)^2 \right\rangle = \langle 1, 2, 6 \rangle \quad \text{(Direction/Tangent Vector)}$$

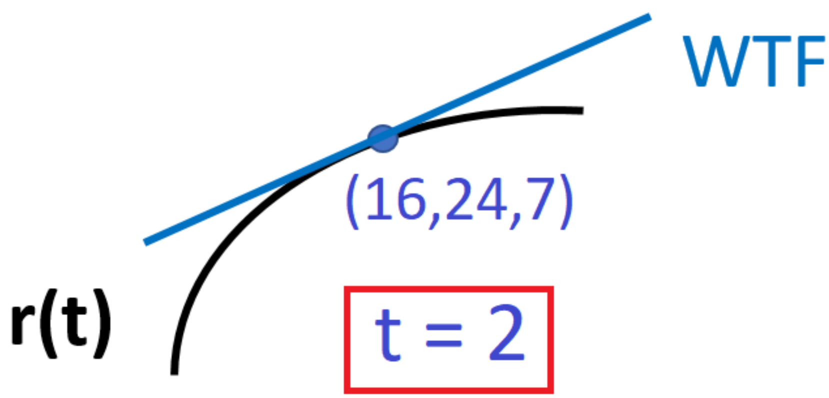
Equation: Point $(2, 1, 4)$, Direction vector $\langle 1, 2, 6 \rangle$, so

$$\begin{cases} x(t) = & 2+t \\ y(t) = & 1+2t \\ z(t) = & 4+6t \end{cases}$$

Also written as $\langle 2+t, 1+2t, 4+6t \rangle$

Example 14: (Good quiz/exam question)

Find the (parametric) equations of the tangent line to the curve $\mathbf{r}(t) = \left\langle (2+t)^2, 3t^3, 4t-1 \right\rangle$ at the point $(16, 24, 7)$



Point: $(16, 24, 7)$

Direction Vector:

$$\mathbf{r}'(t) = \left\langle 2(2+t), 3(3t^2), 4 \right\rangle = \left\langle 4+2t, 9t^2, 4 \right\rangle$$

Find t:

$$\left\langle (2+t)^2, 3t^3, 4t-1 \right\rangle = \langle 16, 24, 7 \rangle$$

The last equation becomes:

$$4t - 1 = 7 \Rightarrow 4t = 8 \Rightarrow t = 2$$

And indeed for $t = 2$ we get $(2+t)^2 = 4^2 = 16$ and $3t^3 = 3(8) = 24$, hence $\boxed{t = 2}$, and we need to calculate:

$$\mathbf{r}'(2) = \left\langle 4 + 2(2), 9(2)^2, 4 \right\rangle = \langle 8, 36, 4 \rangle$$

Equation: Point $(16, 24, 7)$, Direction Vector $\langle 8, 36, 4 \rangle$

$$\begin{cases} x(t) = 16 + 8t \\ y(t) = 24 + 36t \\ z(t) = 7 + 4t \end{cases}$$

Or $\langle 16 + 8t, 24 + 36t, 7 + 4t \rangle$