LECTURE 6: POWER SERIES

Here are a couple more facts about power series:

1. Power Series

Recall: f is **analytic** if f has a power series, that is there are c_n and R such that

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \text{ for all } x \in (-R, R)$$

Fact: If $f(x)$ is analytic then $c_n = \frac{f^{(n)}(0)}{n!}$

Why? If you by set $x = 0$ in the power series, you get $f(0) = c_0$.

Differentiating the power series (ok from last time), we get $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$, and setting $x = 0$ we have $f'(0) = c_1$

Differentiating it again and setting $x = 0$ we get $f''(0) = 2c_2$ so $c_2 = \frac{f''(0)}{2}$ and so on... $\square$

This also shows that power series expansions are unique: If $f(x) = \sum a_n x^n = \sum b_n x^n$ in $(-R, R)$, then $a_n = b_n = \frac{f^{(n)}(0)}{n!}$

A non-analytic smooth function:

$$\text{Let } f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

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In that case can show $c_n = \frac{f^{(n)}(0)}{n!} = 0$ for all n (see homework) so $\sum_{n=0}^{\infty} c_n x^n = 0$ but $f(x)$ is clearly not 0, so f does not equal its power series expansion.

So although analytic functions are smooth (infinitely differentiable), not all smooth functions are analytic.

Finally, using power series, we can define the familiar functions from calculus such as $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\cos(x)$, $\sin(x)$ etc (see Chapter 8 in Rudin if interested)

2. A double sum

In general, if a_{ij} is a doubly-infinite sequence, then

$$\sum_{i=1}^{\infty} \underbrace{\sum_{j=1}^{\infty} a_{ij}}_{\text{Column Sum}} \neq \sum_{j=1}^{\infty} \underbrace{\sum_{i=1}^{\infty} a_{ij}}_{\text{Row Sum}}$$

Non-Example: Let a_{ij} be the matrix with $1's$ on the main diagonal, and $-1's$ on the diagonal above, and 0 everywhere else (see picture in lecture).

Then the column sums are $0, 0, 0, 0, \dots$, so $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = 0$

And the row sums are $1, 0, 0, 0, \dots$, so $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = 1$

That said, under some mild conditions, we can interchange the two sums. It's like a Fubini theorem but for series:

Theorem: [Fubini for series]

$$\text{If } \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}| < \infty \text{ then } \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

The above means that if $b_i =: \sum_{j=1}^{\infty} |a_{ij}|$ and $\sum_i b_i$ converges, then we can interchange the two sums.

In the above example, we got $b_i = 2$ and $\sum_i b_i = \sum 2 = \infty$

Proof: Really elegant!

STEP 1: Let x_n be the sequence $x_n = \frac{1}{n}$ and define the sequence of functions $\{f_i\}_{i=0}^{\infty}$ by:

$$f_i(x_n) = \sum_{j=1}^{n} a_{ij} \text{ (Partial Sum)}$$

$$f_i(0) = \sum_{j=1}^{\infty} a_{ij} \text{ (Series)}$$

$$\text{Define } g(x) = \sum_{i=1}^{\infty} f_i(x),\ x \in \{0, x_1, x_2, \cdots\}$$

Claim # 1: Each f_i is continuous at 0

$$\lim_{n\to\infty} f_i(x_n) \overset{\text{DEF}}{=} \lim_{n\to\infty}\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{\infty} a_{ij} = f_i(0)\checkmark$$

Claim # 2: g is continuous at 0

Notice that for all x, we have

$$|f_i(x)| \leq \sum_{j=1}^{\infty} |a_{ij}| = b_i$$

And since $\sum_{i=1}^{\infty} b_i$ converges, by the Weierstraß M-test, the series defining f_i converges uniformly, and since each $f_i(x)$ is continuous at 0, it follows that g is continuous at 0. ✓

STEP 2: Main Proof

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \overset{\text{DEF}}{=} \sum_{i=1}^{\infty} f_i(0) \overset{DEF}{=} g(0) \overset{\text{CONT}}{=} \lim_{n\to\infty} g(x_n) \overset{\text{DEF}}{=} \lim_{n\to\infty}\sum_{i=1}^{\infty} f_i(x_n) \\
\overset{\text{DEF}}{=} \lim_{n\to\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij} = \lim_{n\to\infty}\sum_{i=1}^{\infty} a_{i1} + a_{i2} + \cdots + a_{in} \\
= \lim_{n\to\infty}\sum_{i=1}^{\infty} a_{i1} + \sum_{i=1}^{\infty} a_{i2} + \cdots + \sum_{i=1}^{\infty} a_{in} \text{ Sum of } n \text{ convergent series} \\
= \lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{\infty} a_{ij} \overset{\text{DEF}}{=} \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}
\end{aligned}$$

3. Taylor's Theorem

Finally, let's answer the famous Taylor question: If f has a power series about $x = 0$, does it have a power series about other points $x = a$ as well?

Yes, provided a is the interval of convergence of f:

Theorem: [Taylor's Theorem]

Suppose $f(x)$ has a power series converging in $(-R, R)$

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

Then, if $a \in (-R, R)$, then f also has a power series about $x = a$

$$f(x) = \sum_{n=0}^{\infty} d_n (x-a)^n$$

This power series converges if $|x - a| < r$, where $r = R - |a|$

Notice: The closer a is to R, the smaller r is (see picture in lecture)

Proof: Surprising application of the previous theorem:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=0}^{\infty} c_n \left((x-a) + a \right)^n \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_n \binom{n}{m} (x-a)^m a^{n-m} \quad \text{(Binomial Theorem)} \\
&\stackrel{?}{=} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} c_n \binom{n}{m} a^{n-m} (x-a)^m \text{ (see picture in lecture)} \\
&= \sum_{m=0}^{\infty} \underbrace{\left(\sum_{n=m}^{\infty} \binom{n}{m} c_n a^{n-m} \right)}_{d_m} (x-a)^m
\end{aligned}$$

So we are done **if** we can justify the interchange of the sums. But for this we can use the Fubini theorem for series, which says it's ok provided that the following series converges

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left|c_n\binom{n}{m}(x-a)^m a^{n-m}\right| \\
=&\sum_{n=0}^{\infty}|c_n|\sum_{m=0}^{n}\binom{n}{m}|x-a|^m|a|^{n-m} \\
=&\sum_{n=0}^{\infty}|c_n|\left(|x-a|+|a|\right)^n \text{ Binomial Theorem}
\end{aligned}
$$

And since the radius of convergence of $\sum c_n x^n$ is R, this converges provided that $|x-a|+|a|<R$ that is $|x-a|<R-|a|=r$, which is what we wanted $\square$

Note: Of course, once the power series exist, we have $d_n=\frac{f^{(n)}(a)}{n!}$ and we get the expansion

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

4. The Gamma Function

The following has nothing to do with series, but is a special function that is used a lot in math and physics, and is especially popular in the YouTube world ☺

Definition:

$$\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt$$

This is defined at least when $x\geq 1$, but can be extended to the case $(0,\infty)$

It is a generalization of the factorial function, mainly because of the following fact:

Fact: For all $x \geq 1$ we have

$$\Gamma(x+1) = x\Gamma(x)$$

Notice how similar this is to the identity $(n+1)! = (n+1)n!$

Proof:

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t} dt \\ &\overset{\text{IBP}}{=} \left[t^x \left(-e^{-t}\right)\right]_0^\infty - \int_0^\infty x t^{x-1} \left(-e^{-t}\right) dt \\ &= 0 + x \int_0^\infty t^{x-1} e^{-t} dt \\ &= x\Gamma(x)\end{aligned}$$

Corollary: $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$

Proof: By induction on n

Base Case:

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t} dt = \left[-e^{-t}\right]_0^\infty = 0 + 1 = 1 = 0! \checkmark$$

Inductive Step:

$$\Gamma(n+2) = (n+1)\Gamma(n+1) = (n+1)n! = (n+1)! \checkmark \quad \square$$

Fact:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Proof:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = 2\int_0^\infty e^{-u^2} du = \int_{-\infty}^\infty e^{-u^2} du = \sqrt{\pi}$$

Here we used the $u-$sub $u = \sqrt{t}$ so $du = \frac{1}{2\sqrt{t}} dt$, as well as the famous Gaussian Integral

The Gamma function is useful in the evaluation of some integrals, see Rudin for examples. Here is an even cooler application:

5. Optional: The Half Derivative

Video: Half Derivative

Question: What is the half derivative of x?

More precisely we would like an operation $D^{\frac{1}{2}}$ in such a way that

$$D^{\frac{1}{2}}\left(D^{\frac{1}{2}} x\right) = Dx = 1$$

Notice:

$$\begin{aligned} Dx^n &= nx^{n-1} \\ D^2 x^n &= n(n-1)x^{n-2} = \frac{n!}{(n-2)!} x^{n-2} \\ D^k x^n &= \frac{n!}{(n-k)!} x^{n-k} \end{aligned}$$

So by analogy, we should define

$$D^{\frac{1}{2}}x^n = \frac{n!}{\left(n-\frac{1}{2}\right)!}x^{n-\frac{1}{2}}$$

Since factorials are not defined for non-integers, we need to use the Gamma function, and in fact we get

Definition:

$$D^{\frac{1}{2}}x^n = \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}x^{n-\frac{1}{2}}$$

This is valid even if n is not an integer, and you can check that this definition does its job:

Fact:

$$D^{\frac{1}{2}}\left(D^{\frac{1}{2}}x^n\right) = nx^{n-1}$$

In the case $n = 1$ we get the more explicit formula:

$$D^{\frac{1}{2}}x = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)}x^{\frac{1}{2}}$$

$$\begin{aligned} \Gamma(2) &= 1! = 1 \\ \Gamma\left(\frac{3}{2}\right) &= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2} \\ \text{Hence } D^{\frac{1}{2}}x &= \frac{2}{\sqrt{\pi}}\sqrt{x} = 2\sqrt{\frac{x}{\pi}} \end{aligned}$$

Applications: Fractional derivatives are used in physics to describe broken processes. There are also fractional differential equations in PDE. Check out this playlist if you want to learn more about fractional derivatives.