LECTURE 7: INFINITE LIMITS, MONOTONE SEQUENCES

1. Example 8: Exponential Limit

Video: Limit Example 8: Exponential

Let's continue our exploration of exponential limits!

Example 8:

$$\text{If } |a| < 1, \text{ then } \lim_{n\to\infty} a^n = 0$$

Proof: Assume $a \neq 0$ (the case $a = 0$ is trivial)

STEP 1: Scratch work:

$$|s_n - s| = |a^n - 0| = |a|^n$$

Clever Observation: Since $|a| < 1$, we can write $|a| = \frac{1}{1+b}$ for some $b > 0$, namely $b = \frac{1}{|a|} - 1$.

(Example: If $|a| = \frac{2}{3}$, then $\frac{2}{3} = \frac{1}{\frac{3}{2}} = \frac{1}{1+\frac{1}{2}}$, so $b = \frac{1}{2}$).

Hence $|a|^n = \frac{1}{(1+b)^n}$

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But by the binomial theorem:

$$(1+b)^n = 1^n + n1^{n-1}b + \text{ Positive terms } \geq 1 + nb > nb$$

$$\text{Hence } |a|^n = \frac{1}{(1+b)^n} < \frac{1}{nb} < \epsilon \Rightarrow \frac{1}{n} < b\epsilon \Rightarrow n > \frac{1}{b\epsilon}$$

Which suggests to use $N = \frac{1}{b\epsilon}$

STEP 2: Let $\epsilon > 0$ be given, let $N = \frac{1}{b\epsilon}$, then if $n > N$, we have

$$|a^n - 0| = |a|^n = \frac{1}{(b+1)^n} < \frac{1}{nb} < \frac{1}{b}\left(\frac{1}{n}\right) < \frac{1}{b}(b\epsilon) = \epsilon\checkmark$$

Hence $\lim_{n\to\infty} a^n = 0$ $\square$

2. Example 9: Don't use L'Hôpital

Video: Limit Example 9: $n^{\frac{1}{n}}$

Example 9:

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Proof: Let $s_n = n^{\frac{1}{n}} - 1$ and show $s_n \to 0$.

The idea is to use the Squeeze Theorem. On the one hand, since $n \geq 1 \Rightarrow n^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1$, so $s_n = n^{\frac{1}{n}} - 1 \geq 0$ (So our lower function is 0).

On the other hand, notice $1 + s_n = n^{\frac{1}{n}}$, so $(1+s_n)^n = n$, and therefore, by the binomial theorem (this time with more terms):

$$\begin{aligned} n =&(1+s_n)^n \\ =&1^n + n1^{n-1}s_n + \frac{n(n-1)}{2}1^{n-2}(s_n)^2 + \text{ Positive terms} \\ =&1 + ns_n + \frac{n(n-1)}{2}(s_n)^2 + \text{ Positive terms} \\ >&\frac{n(n-1)}{2}(s_n)^2 \end{aligned}$$

$$\begin{aligned} \text{Hence } \frac{n(n-1)}{2}\left(s_n\right)^2 &< n \\ \Rightarrow& (n-1)(s_n)^2 < 2 \\ \Rightarrow& s_n < \sqrt{\frac{2}{n-1}} \end{aligned}$$

Therefore we get $0 \leq s_n < \sqrt{\frac{2}{n-1}}$ and a standard squeeze theorem argument shows that $s_n = n^{\frac{1}{n}} - 1 \to 0$, hence $n^{\frac{1}{n}} \to 1$ $\square$

Corollary:

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1 \qquad (\text{If } a > 0)$$

Example: $2^{\frac{1}{n}} \to 1$.

Proof:

Case 1: $a \geq 1$

Then, on the one hand, $a^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1$, but on the other hand, if n is large (more specifically $n \geq a$), then $n^{\frac{1}{n}} \geq a^{\frac{1}{n}}$ and so $1 \leq a^{\frac{1}{n}} \leq n^{\frac{1}{n}}$, and the squeeze theorem (and Example 9) takes care of the rest.

Case 2: $0 < a < 1$

$$\text{Then } \lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{\frac{1}{a}}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\left(\frac{1}{a}\right)^{\frac{1}{n}}} = \frac{1}{\lim_{n\to\infty}\left(\frac{1}{a}\right)^{\frac{1}{n}}} = \frac{1}{1} = 1\checkmark \quad \square$$

In the last step we used $\frac{1}{a} > 1$ and so $\lim_{n\to\infty}\left(\frac{1}{a}\right)^{\frac{1}{n}} = 1$ (by Case 1) $\square$.

3. Infinite Limits

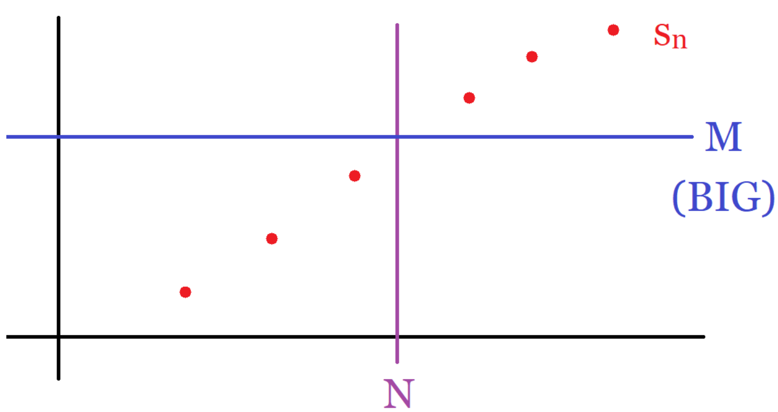
Video: Limit Example 10: Infinite Limits

Finally, what does it mean for a sequence s_n to go to ∞? It just means that s_n eventually becomes bigger than any number we want.

Definition:

(a) $\lim_{n\to\infty} s_n = \infty$ means:

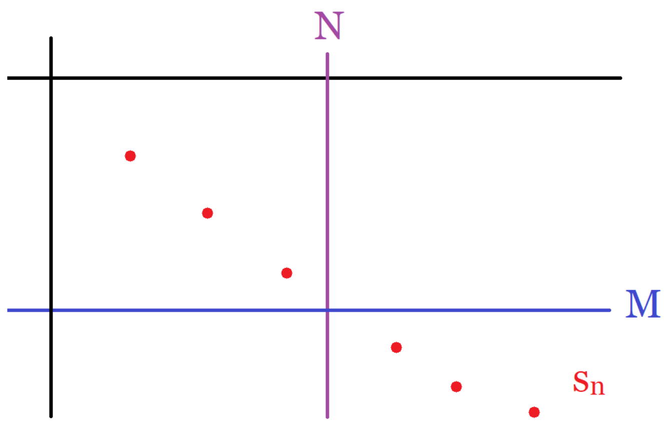
For all M there is N such that if $n > N$, then $s_n > M$



Definition:

(b) $\lim_{n\to\infty} s_n = -\infty$ means:

For all M there is N such that if $n > N$, then $s_n < M$



Example 10:

$$\lim_{n\to\infty} \sqrt{n-2} + 3 = \infty$$

STEP 1: Find N

$\sqrt{n-2}+3 > M \Rightarrow \sqrt{n-2} > M-3 \Rightarrow n-2 > (M-3)^2 \Rightarrow n > (M-3)^2+2$

This suggests to let $N = (M-3)^2 + 2$

STEP 2: Let $M > 0$ be given, let $N = (M-3)^2 + 2$, then if $n > N$, we have (assume $M > 3$)

$\sqrt{n-2}+3 > \sqrt{(M-3)^2+2-2}+3 = \sqrt{(M-3)^2}+3 = M-3+3 = M\checkmark$

Hence $\lim_{n\to\infty} \sqrt{n-2} + 3 = \infty$ $\square$

4. Some Infinite Limit Laws

Video: Infinite Limit Laws

We can also prove limit laws for infinite limits:

Fact:

Suppose $\lim_{n\to\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n = t > 0$ (or $t = \infty$) then

$$\lim_{n\to\infty} s_n t_n = \infty$$

(In other words, $\infty \times t = \infty$)

STEP 1: Scratchwork

In either case ($t > 0$ or $t = \infty$), if n is large enough, then $t_n > m$ for some $m > 0$

Hence $s_n t_n \geq m s_n$ and this is $> M$ if $s_n > \frac{M}{m}$.

STEP 2: Let $M > 0$ be given.

Then there is $m > 0$ and N_1 such that if $n > N_1$, then $t_n > m$.

Now since $s_n \to \infty$, there is N_2 such that if $n > N_2$, then $s_n > \frac{M}{m}$.

Let $N = \max\{N_1, N_2\}$, then if $n > N$, we have:

$$s_n t_n \geq s_n(m) > \left(\frac{M}{m}\right) m = M \checkmark$$

Hence $\lim_{n\to\infty} s_n t_n = \infty$.

5. Duality

Video: Duality Theorem

Finally, the following theorem illustrates the beautiful relationship between finite and infinite limits.

Duality Theorem:

If $s_n > 0$ for all n, then

$$\lim_{n\to\infty} s_n = \infty \Leftrightarrow \lim_{n\to\infty} \frac{1}{s_n} = 0$$

Example: We know $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$ (Example 8 with $a = \frac{1}{2}$), hence

$$\lim_{n\to\infty} \frac{1}{\left(\frac{1}{2}\right)^n} = \infty \quad \text{so} \quad \lim_{n\to\infty} 2^n = \infty$$

Proof:

($\Rightarrow$) Let $\epsilon > 0$ be given

Let $M = \frac{1}{\epsilon} > 0$. Since $s_n \to \infty$, there is N such that for all $n > N$, $s_n > M = \frac{1}{\epsilon}$.

But then for that same N, if $n > N$, we have:

$$\left|\frac{1}{s_n} - 0\right| = \frac{1}{|s_n|} = \frac{1}{s_n} < \frac{1}{M} = \frac{1}{\frac{1}{\epsilon}} = \epsilon\checkmark$$

Hence $\lim_{n\to\infty} \frac{1}{s_n} = 0$

($\Leftarrow$) Let $M > 0$ be given

Let $\epsilon = \frac{1}{M}$. Then since $\frac{1}{s_n} \to 0$ there is N such that for all $n > N$, $\left|\frac{1}{s_n}\right| = \frac{1}{s_n} < \epsilon = \frac{1}{M}$.

But then, for that same N, if $n > N$, we have:

$$s_n > \frac{1}{\epsilon} = \frac{1}{\frac{1}{M}} = M \checkmark$$

Hence $\lim_{n\to\infty} s_n = \infty$. $\square$

Isn't this proof elegant? This is Analysis at its finest! ☺

6. Monotone Sequence Theorem

Video: Monotone Sequence Theorem

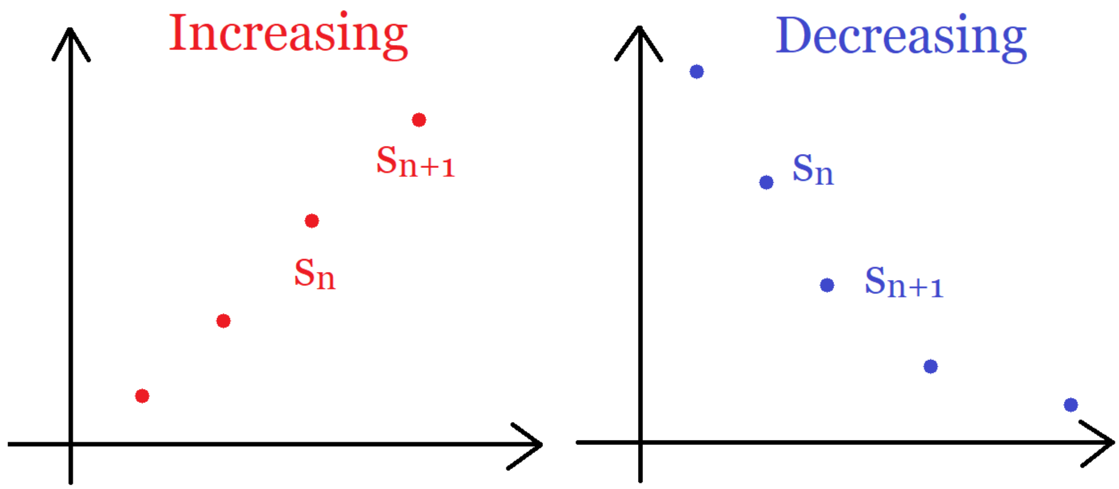
Notice how *annoying* it is to show that a sequence explicitly converges, and it would be nice if we had some easy general theorems that guarantee that a sequence converges.

Definition:

(s_n) is **increasing** if $s_{n+1} > s_n$ for all n

(s_n) is **decreasing** if $s_{n+1} < s_n$ for all n

If either of the above holds, we say that (s_n) is **monotonic**.



Examples: $s_n = \sqrt{n}$ is increasing, $s_n = \frac{1}{n}$ is decreasing, $s_n = (-1)^n$ is neither increasing nor decreasing.

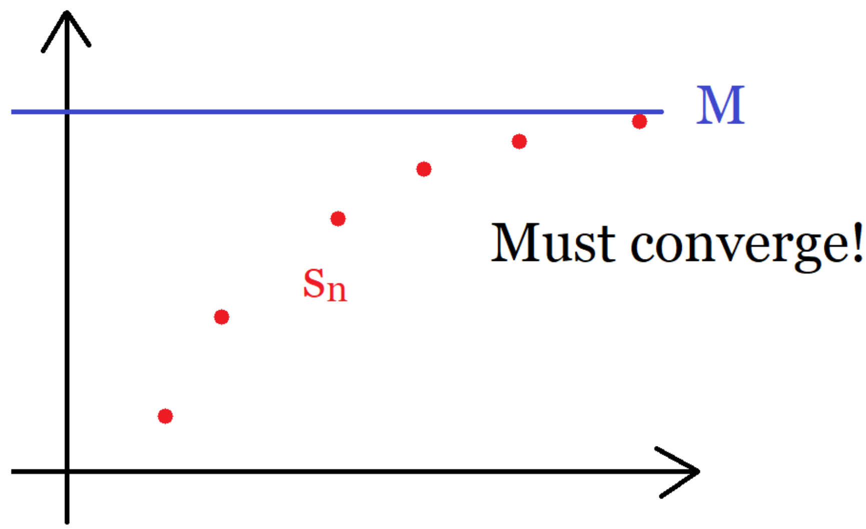
The following theorem gives a **very** elegant criterion for a sequence to converge, and explains why monotonicity is so important.

(Important) Monotone Sequence Theorem:

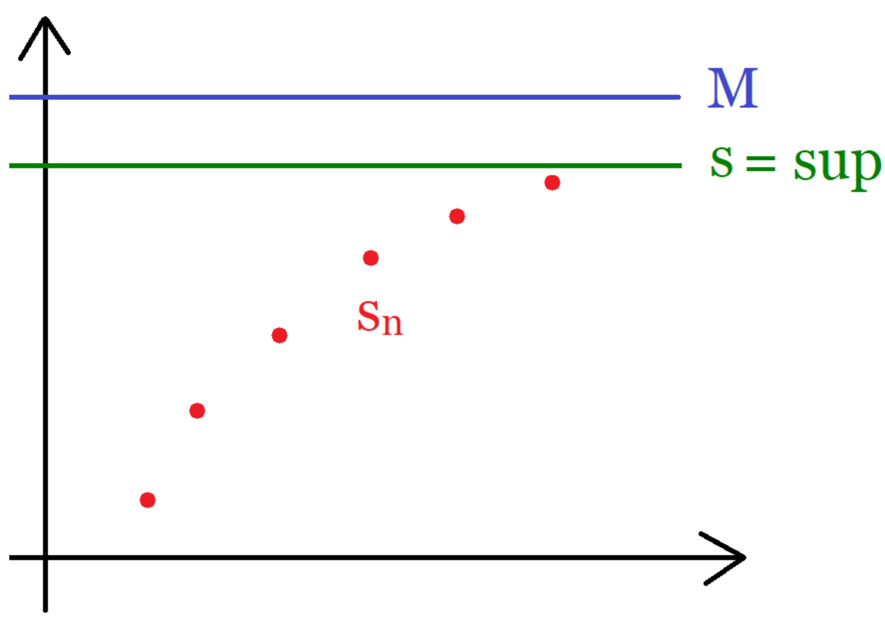
(s_n) is increasing and bounded above, then (s_n) converges.

Note: The same proof works if (s_n) is nondecreasing ($s_{n+1} \geq s_n$)

Intuitively: If (s_n) is increasing and has a ceiling, then there's no way it cannot converge. In fact, try drawing a counterexample, and you'll see that it doesn't work!



WARNING: Just because (s_n) is bounded above by M, this does **NOT** imply that s_n converges to M, as the following picture shows. But what *is* true is that s_n converges to the sup of all the s_n



Proof: Elegant interplay between sup (section 4) and convergence (section 8)

STEP 1: Consider

$$S =: \{s_n \mid n \in \mathbb{N}\}$$

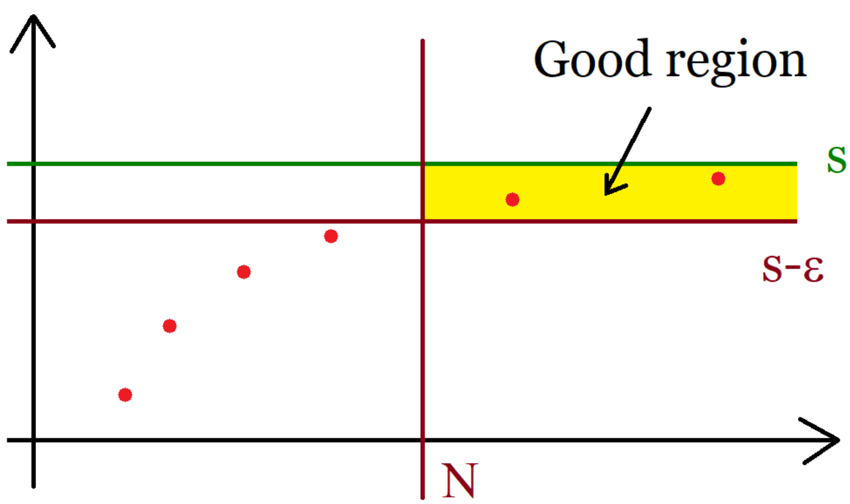
Since $s_n \leq M$ for all M, S is bounded above, hence S has a least upper bound $\boxed{s =: \sup(S)}$

Claim: $\lim_{n \to \infty} s_n = s$.

STEP 2: Let $\epsilon > 0$ be given.

We need to find N such that if $n > N$, then $|s_n - s| < \epsilon$.

Consider $s - \epsilon < s$. By definition of a sup, this means that there is $s_N \in S$ such that $s_N > s - \epsilon$



But then, for that N, if $n > N$, since (s_n) is increasing, we have

$$s_n - s > s_N - s > -\epsilon \Rightarrow s_n - s > -\epsilon$$

On the other hand, since $s = \sup(S)$, we have $s_n \leq s$ for all s and so

$$s_n - s \leq s - s = 0 < \epsilon \Rightarrow s_n - s < \epsilon$$

Combining, we get:

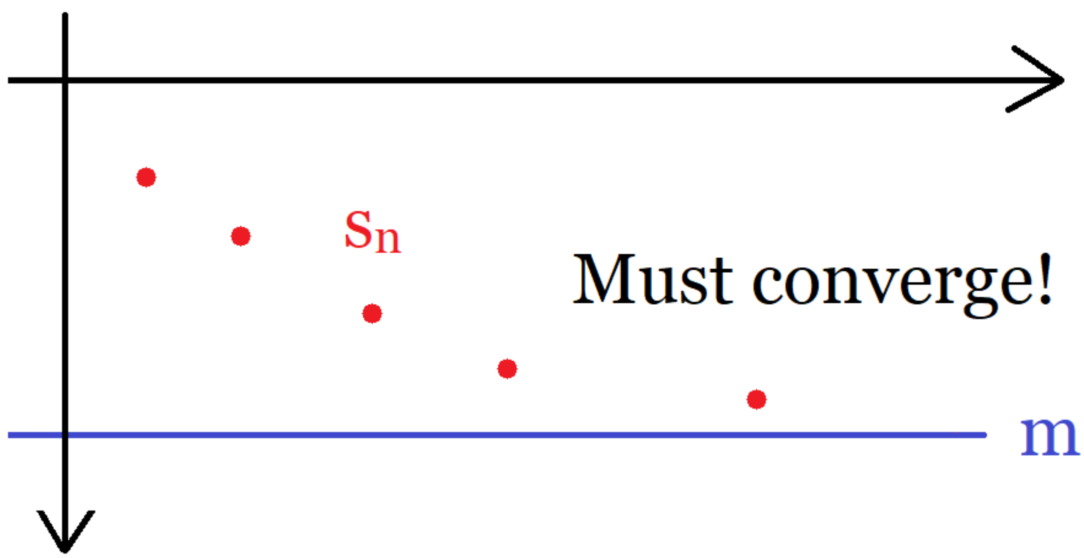
$$-\epsilon < s_n - s < \epsilon \Rightarrow |s_n - s| < \epsilon$$

And so (s_n) converges to s $\square$

By considering $-s_n$ we get the following corollary:

Corollary:

(s_n) is decreasing and bounded below, then (s_n) converges.



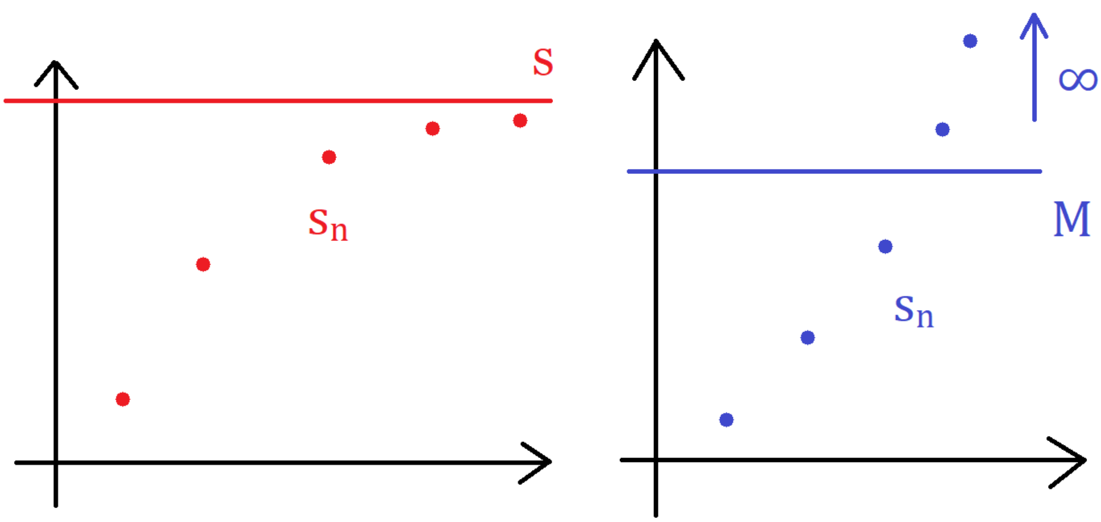
Why? In that case $(-s_n)$ is increasing and bounded above, so converges to some s, and therefore (s_n) converges to $-s$ (or repeat the above proof, but with inf) $\square$

In fact: We don't even need (s_n) to be bounded above, provided that we allow ∞ as a limit.

Theorem:

(s_n) is increasing, then it either converges or goes to ∞

So there are really just 2 kinds of increasing sequences: Either those that converge or those that blow up to ∞.

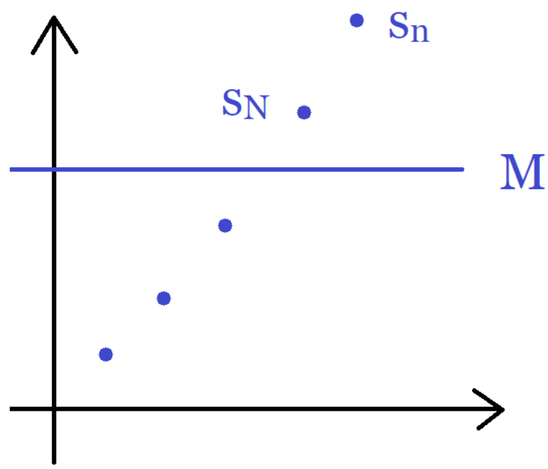


Proof:

Case 1: (s_n) is bounded above, but then by the Monotone Sequence Theorem, (s_n) converges ✓

Case 2: (s_n) is not bounded above, and we claim that $\lim_{n\to\infty} s_n = \infty$.

Let $M > 0$ be given, want to find N such that if $n > N$, then $s_n > M$.



First, there must be N such that $s_N > M$, because otherwise $s_N \leq M$ for all N and so M would be an upper bound for (s_n) $\Rightarrow\Leftarrow$

With that N, if $n > N$, then since (s_n) is increasing, we get $s_n > s_N = M$, so $s_n > M$ and hence s_n goes to ∞ ✓ $\square$

Finally, notice that the proof of the Monotone Sequence Theorem uses the Least-Upper Bound Property (because we defined sup), but in fact something even more amazing is true:

Cool Fact:

The Least Upper Bound Property is **equivalent** to the Monotone Sequence Theorem! (WOW)

7. Decimal Expansions (optional)

Video: Decimal Expansions

As an application of the Monotone Sequence Theorem, we can construct the real numbers via decimal expansions.

Motivation: What does is mean for $\pi = 3.1415\cdots$?

Notice:

$$\begin{aligned}\pi =& 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \cdots \\ =& 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \cdots \\ =& k + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots\end{aligned}$$

Now consider the following sequence (s_n)

$$\begin{aligned}s_0 =& 3 = k \\ s_1 =& 3.1 = k + \frac{d_1}{10} \\ s_2 =& 3.14 = k + \frac{d_1}{10} + \frac{d_2}{10^2} \\ s_3 =& 3.141 = k + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \\ s_n =& 3.1415\cdots d_n = k + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}\end{aligned}$$

Notice that (s_n) is bounded above by $4 = k + 1$ and moreover, (s_n) is increasing (since we're only adding positive terms), therefore by the monotone sequence theorem, (s_n) is converging to s, and this limit is what we call

$$\pi = 3.1415\cdots = k.d_1d_2d_3\cdots$$

So, in some sense, it is reasonable to define real numbers as follows:

Definition:

$\mathbb{R}$ is the set of all numbers of the form

$$k.d_1d_2\cdots$$

Where $k \in \mathbb{Z}$ and each d_i is a digit between 0 and 9 (and $\cdots$ is to be understood in the limit sense as above)

Of course, this leaves many questions to be unanswered, such as: "Does *every* real number (such as $\sqrt{2}$) even have a decimal expansion?" or "How can you show that a rational number is a real number?" Those questions are answered in section 16 (which we won't cover)

More importantly, how would you show that $\mathbb{R}$ (as constructed above) has the least-upper bound property?

There is actually a small glitch in the above definition. For this we need the following formula, which you might remember from calculus (for a proof, see exercise 9.18):

Geometric Series:

If $|r| < 1$, then

$$\lim_{n\to\infty} 1 + r + r^2 + \cdots + r^n = \frac{1}{1-r}$$

$$
\begin{aligned}
0.99999\cdots =& \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots \\
=& \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right) \\
=& \frac{9}{10}\left(\frac{1}{1 - \frac{1}{10}}\right) \\
=& \frac{9}{10}\left(\frac{1}{\frac{9}{10}}\right) \\
=& 1
\end{aligned}
$$

So actually we have $0.999999\cdots = 1.00000\cdots$, so both of those decimal expansions actually represent the same real number! So the above construction is bad in the sense that different decimal expansions might give you the same number.

There is an easy way to get around that, actually: In the above construction, simply throw away decimal expansions that end with an infinite string of $9's$. That is, in the above definition, consider just the decimal expansions that don't end with $9's$.