LECTURE 7: VECTOR FUNCTIONS (II)

Welcome to the second part of our vector function extravaganza, and let's continue with tangent vectors:

1. Unit Tangent Vector

Sometimes it's useful to have the tangent vector have length 1. This is called the **unit** tangent vector:

Example 1:

Find the **unit** tangent vector $\mathbf{T}(t)$ to the curve $\mathbf{r}(t) = \left\langle t, t^2, 2t^2 \right\rangle$

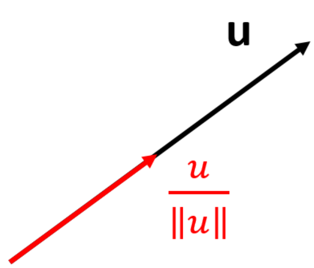
First find the tangent vector:

$$\mathbf{r}'(t) = \langle 1, 2t, 4t \rangle$$

Unit just means "Has Length 1"

Recall:

For any vector $\mathbf{u}$, $\frac{\mathbf{u}}{\|\mathbf{u}\|}$ has length 1



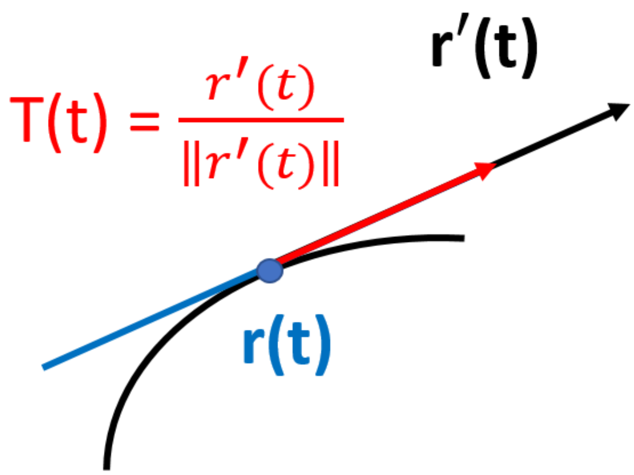
Date: Monday, September 13, 2021.

Definition:

The **unit** tangent vector to $\mathbf{r}(t)$ is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

(Same direction as $\mathbf{r}'(t)$, but length 1)



$$\begin{aligned}\mathbf{r}'(t) &= \langle 1, 2t, 4t\rangle \\ \|\mathbf{r}'(t)\| &= \sqrt{1 + (2t)^2 + (4t)^2} = \sqrt{1 + 4t^2 + 16t^2} = \sqrt{1 + 20t^2}\end{aligned}$$

And therefore:

$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\ &= \frac{1}{\sqrt{1 + 20t^2}}\langle 1, 2t, 4t\rangle \\ &= \left\langle \frac{1}{\sqrt{1 + 20t^2}}, \frac{2t}{\sqrt{1 + 20t^2}}, \frac{4t}{\sqrt{1 + 20t^2}} \right\rangle\end{aligned}$$

(And in fact you can check that $\mathbf{T}(t)$ has length 1)

Note: It is not *immediately* apparent why this is useful, but there are some theorems where you **have** to use the unit tangent vector in order to make the formulas work.

Example 2:

Find $\mathbf{T}(0)$, where $\mathbf{r}(t) = \left\langle 1, e^t, e^{2t} \right\rangle$

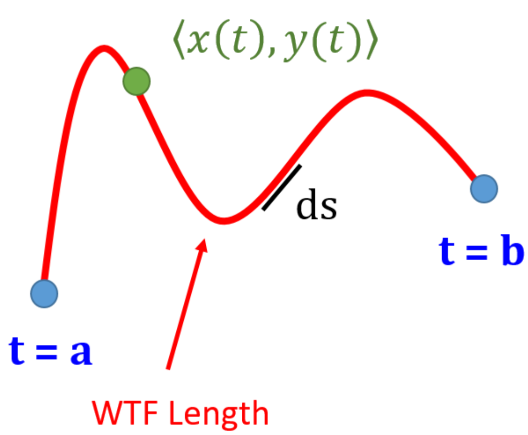
Here it is a bit easier than the previous problem:

$$\mathbf{r}'(t) = \left\langle 0, e^t, 2e^{2t} \right\rangle$$
$$\mathbf{r}'(0) = \left\langle 0, e^0, 2e^{2(0)} \right\rangle = \langle 0, 1, 2 \rangle$$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\langle 0,1,2\rangle}{\|\langle 0,1,2\rangle\|} = \frac{1}{\sqrt{5}}\langle 0,1,2\rangle = \left\langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

2. Arclength

You can also use vector equations to calculate the length of a curve.



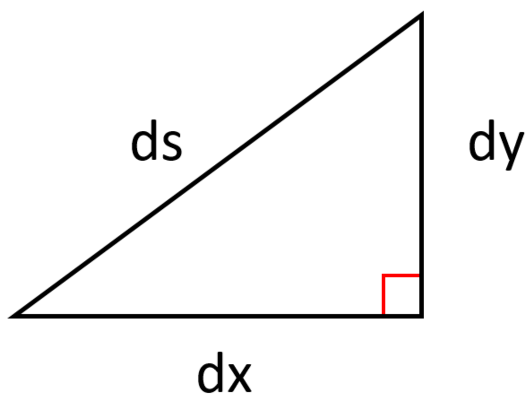
Fact:

The length of a curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ from $t = a$ to $t = b$ is

$$\text{Length } = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

Why?

STEP 1: Approximate the curve with little segments of length ds (as in the figure above).



STEP 2: By the Pythagorean Theorem, the length of each little segment is

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

Where dx is a small change in x and dy is a small change in y

STEP 3: However, by the chain rule, we have

$$dx = \left(\frac{dx}{dt}\right) dt = x'(t)dt$$
$$dy = \left(\frac{dy}{dt}\right) dt = y'(t)dt$$

And therefore, we get:

$$\begin{aligned} ds =& \sqrt{(dx)^2 + (dy)^2} \\ =& \sqrt{(x'(t)dt)^2 + (y'(t)dt)^2} \\ =& \sqrt{(x'(t))^2(dt)^2 + (y'(t))^2(dt)^2} \\ =& \sqrt{(x'(t))^2 + (y'(t))^2}\sqrt{(dt)^2} \\ =& \sqrt{(x'(t))^2 + (y'(t))^2}dt \end{aligned}$$

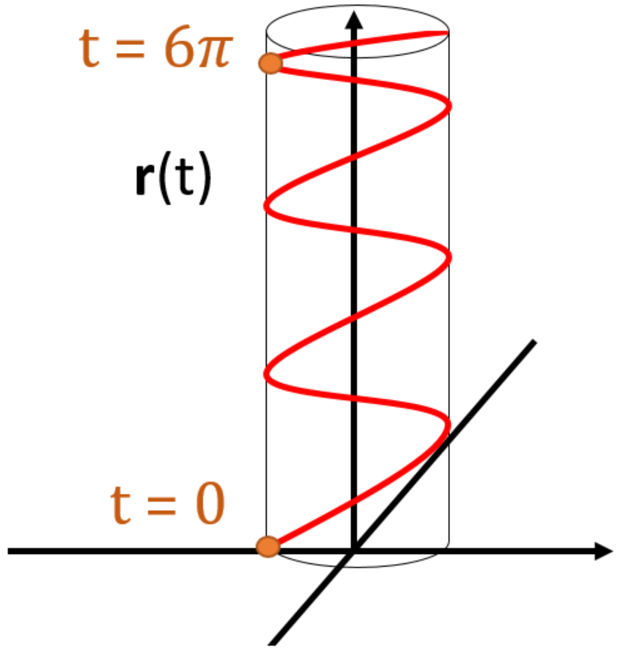
STEP 4: Finally, to get the length of the curve, sum up/integrate the little segments to get:

$$\text{Length } = \int_a^b ds = \int_a^b \sqrt{(dx)^2 + (dy)^2} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt \quad \square$$

In 3 dimensions it's exactly the same:

Example 3: (Good Quiz/Exam Question)

Find the length of the helix $\mathbf{r}(t) = \langle 2\cos(5t), 2\sin(5t), 3t\rangle$ from $t = 0$ to $t = 6\pi$



Fact:

$$\text{Length} = \int_0^{6\pi} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

Here this becomes:

$$\begin{aligned}
\text{Length} &= \int_0^{6\pi} \sqrt{(-2\sin(5t)(5))^2 + (2\cos(5t)(5))^2 + 3^2}\,dt \\
&= \int_0^{6\pi} \sqrt{(-10)^2\sin^2(5t) + (10)^2\cos^2(5t) + 9}\,dt \\
&= \int_0^{6\pi} \sqrt{100\left(\sin^2(5t) + \cos^2(5t)\right) + 9}\,dt \\
&= \int_0^{6\pi} \sqrt{100 + 9}\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{6\pi} \sqrt{109} dt \\
&= \sqrt{109}(6\pi - 0) \\
&= 6\pi\sqrt{109}
\end{aligned}
$$

Example 4: (Good Quiz/Exam Question)

Find the length of the curve $\mathbf{r}(t) = \left\langle \frac{1}{2}e^{2t}, 2e^t, 2t \right\rangle$ from $t = 0$ to $t = 3$

$$
\begin{aligned}
\text{Length } &= \int_0^3 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt \\
&= \int_0^3 \sqrt{\left(\frac{1}{2}2e^{2t}\right)^2 + (2e^t)^2 + 2^2} dt \\
&= \int_0^3 \sqrt{e^{4t} + 4e^{2t} + 4} dt \\
&= \int_0^3 \sqrt{\left(e^{2t}\right)^2 + 2\left(e^{2t}\right)(2) + 2^2} dt \\
&= \int_0^3 \sqrt{\left(e^{2t} + 2\right)^2} dt \\
&= \int_0^3 e^{2t} + 2 dt \\
&= \left[\frac{1}{2}e^{2t} + 2t\right]_0^3 \\
&= \frac{1}{2}e^6 + 2(3) - \frac{1}{2}e^0 - 2(0) \\
&= \frac{e^6}{2} + \frac{11}{2}
\end{aligned}
$$

3. Dot and Cross Products

One **new** thing we can do with vector functions is take dot products and cross products (since they're vectors after all)

Example 5:

Find $(\mathbf{r} \cdot \mathbf{s})(t)$ and $(\mathbf{r} \times \mathbf{s})(t)$, where $\mathbf{r}(t) = \langle 1, t, 2t \rangle$ and $\mathbf{s}(t) = \left\langle t, 2t^2, 3t \right\rangle$

$$
\begin{aligned}
(\mathbf{r} \cdot \mathbf{s})(t) \overset{DEF}{=} & \mathbf{r}(t) \cdot \mathbf{s}(t) \\
= & \langle 1, t, 2t \rangle \cdot \left\langle t, 2t^2, 3t \right\rangle \\
= & (1)(t) + (t)\left(2t^2\right) + (2t)(3t) \\
= & t + 2t^3 + 6t^2
\end{aligned}
$$

Notice this is a *scalar* function and $(\mathbf{r} \cdot \mathbf{s})'(t) = 1 + 6t^2 + 12t$

The nice thing is that the product rule (Prada Lu) is true for dot products:

Dot Product Rule:

$$(\mathbf{r} \cdot \mathbf{s})'(t) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$$

For the cross product it's similar:

$$
\begin{aligned}
(\mathbf{r}\times\mathbf{s})(t) &\overset{\text{DEF}}{=} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & t & 2t \\ t & 2t^2 & 3t \end{vmatrix} \\
&= \begin{vmatrix} t & 2t \\ 2t^2 & 3t \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2t \\ t & 3t \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & t \\ t & 2t^2 \end{vmatrix}\mathbf{k} \\
&= \left(3t^2 - 4t^3\right)\mathbf{i} - \left(3t - 2t^2\right)\mathbf{j} + \left(2t^2 - t^2\right)\mathbf{k} \\
&= \left\langle 3t^2 - 4t^3, -3t + 2t^2, t^2 \right\rangle
\end{aligned}
$$

Here the cross product is a *vector* function. And here the nice thing again is that the Prada Lu is true for cross products:

Cross Product Rule:

$$(\mathbf{r}\times\mathbf{s})'(t) = \mathbf{r}'(t)\times\mathbf{s}(t) + \mathbf{r}(t)\times\mathbf{s}'(t)$$

Here's a small exercise with an interesting geometric interpretation:

Example 6:

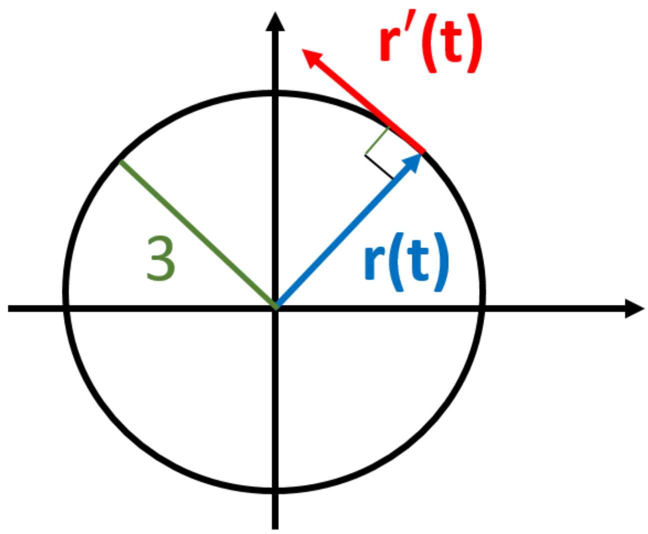
Suppose $\|\mathbf{r}(t)\| = 3$ (or any constant), show that $\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$

(I'll tell you the geometric interpretation of this at the end)

Trick: Notice:

$$
\begin{aligned}
\|\mathbf{r}(t)\|^2 =& 3^2 \\
\mathbf{r}(t) \cdot \mathbf{r}(t) =& 9 \\
(\mathbf{r}(t) \cdot \mathbf{r}(t))' =& (9)' = 0 \\
\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) =& 0 \quad \text{(Product Rule)} \\
2\mathbf{r}'(t) \cdot \mathbf{r}(t) =& 0 \\
\mathbf{r}'(t) \cdot \mathbf{r}(t) =& 0
\end{aligned}
$$

Interpretation: $\|\mathbf{r}(t)\| = 3$ implies that $\mathbf{r}(t)$ lies on a circle (or sphere) centered at the origin and radius 3, so $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$ is saying that the tangent vector of a circle is perpendicular to its radius:



This is a fact known from geometry, and here we proved it quite elegantly with vector functions!