LECTURE 7: FOURIER SERIES

Welcome to the magical world of Fourier series! It is an important sub-field of Analysis with lots of applications to physics and PDE. [1]

1. Introduction

Main Goal: Given a 2π periodic function f, write f as a **trigonometric series**, that is in the form

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Where a_n and b_n are complex numbers ($b_0 = 0$ by convention)

It's like a power series, except with cos and sin instead of x^n

First Observation: Because $e^{inx} = \cos(nx) + i\sin(nx)$ and the a_n and b_n are arbitrary, we can write this more compactly as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Where c_n are complex numbers

A formula for c_n

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[1]The presentation follows the book *Fourier Analysis: An Introduction* by Stein and Shakarchi, and goes a bit deeper than Chapter 8 in Rudin

Fix m, multiply f by e^{-imx} and *formally* integrate on $[-\pi, \pi]$ to get

$$\int_{-\pi}^{\pi} f(x)e^{-imx}dx = \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} c_n e^{inx} \right) e^{-imx}dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}dx$$

But if $n \neq m$ we get

$$\begin{aligned}\int_{-\pi}^{\pi} e^{i(n-m)x}dx &= \left[\frac{e^{i(n-m)x}}{i(n-m)}\right]_{-\pi}^{\pi} = \frac{1}{i(n-m)} \left(e^{i(n-m)\pi} - e^{i(n-m)(-\pi)}\right) \\ &= \frac{1}{i(n-m)} \Big(\cos((n-m)\pi) + i\sin((n-m)\pi)^{\to 0} \\ &\quad - \cos((n-m)(-\pi)) - i\sin((n-m)(-\pi))^{\to 0} \Big) \\ &= 0\end{aligned}$$

(Here we used the fact that cos is even)

And if $n = m$ then we get

$$\int_{-\pi}^{\pi} e^{i(m-m)x}dx = \int_{-\pi}^{\pi} 1dx = 2\pi$$

So in the end, the sum above just becomes

$$\int_{-\pi}^{\pi} f(x)e^{-imx}dx = 2\pi c_m \Rightarrow c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-imx}dx$$

Which gives us an explicit formula for c_n

Fact: If $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ on $[-\pi, \pi]$, then (at least formally)

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

This is an an analog of $\frac{f^{(n)}(0)}{n!}$ but for trig series.

Because of this, it makes sense to define

Definition: [n-th Fourier coefficient]

$$\hat{f}(n) =: \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

Example: Consider $f(x) = x$ on $[-\pi, \pi]$

Using an integration by parts, you can show that if $n \neq 0$, then

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{(-1)^{n+1}}{in}$$

$$\text{and } \hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx = 0$$

Therefore the Fourier series of f is

$$\sum_{n\neq 0} \frac{(-1)^{n+1}}{in}e^{inx} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx)$$

Were we have used that $\frac{e^{inx}-e^{-inx}}{2i} = \sin(nx)$

To study Fourier series more rigorously, we need partial sums.

Definition: The N-th **partial sum** of the Fourier series of f on $(-\pi, \pi)$ is

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \qquad \text{(Trig Polynomial)}$$

Main Goal: In what sense does $S_N(f)$ converge to f as $N \to \infty$?

Remark: Here we mainly focus on the interval $[-\pi, \pi]$. The same thing works on other intervals. For example, if f is periodic of period 1, then on $[0, 1]$ we get

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$$

Here instead of $\frac{1}{2\pi}$ we have $\frac{1}{1} = 1$

2. Uniqueness

Let's now show that, in some sense, the Fourier series is unique:

Theorem: [Uniqueness]

Let f be a (bounded) integrable 2π periodic function with $\hat{f}(n) = 0$ for all n. If f is continuous at x_0 then $f(x_0) = 0$

(In particular, if f is continuous everywhere, then we get $f = 0$)

Proof:[2] For simplicity, assume $x_0 = 0$ and f is real-valued

STEP 1: By contradiction, suppose $\hat{f}(n) = 0$ for all n but $f(0) \neq 0$. WLOG, $f(0) > 0$.

[2]The proof is taken from Theorem 2.1 in Stein and Shakarchi

Main Idea: Construct a sequence p_n of trig polynomials that "peak" at 0, in the sense that $\int p_n(x)f(x)dx \to \infty$. This will be a contradiction because, by assumption on $\hat{f}(n)$, those integrals are 0 for all n (see below)

Since f is continuous at 0, with $\epsilon = \frac{f(0)}{2}$ there is $\delta > 0$ such that if $|x| < \delta$ then

$$|f(x) - f(0)| < \frac{f(0)}{2} \Rightarrow f(x) - f(0) > -\frac{f(0)}{2} \Rightarrow f(x) > \frac{f(0)}{2}$$

STEP 2: Define $p(x) = \epsilon + \cos(x)$, where $\epsilon > 0$ is so small that

$$|p(x)| \leq 1 - \frac{\epsilon}{2} \text{ on } |x| \geq \delta$$

(Something like $\epsilon = \frac{2}{3}(1 - \cos(\delta))$ should work, at least around 0)

And choose $\eta > 0$ smaller than δ such that

$$|p(x)| \geq 1 + \frac{\epsilon}{2} \text{ on } |x| \leq \eta$$

We can do that because $p(0) = 1 + \epsilon$ and p is continuous

Finally define our desired sequence p_n as

$$p_n(x) =: (p(x))^n = (\epsilon + \cos(x))^n$$

Since p_n is a trigonometric polynomial and all the Fourier coefficients are zero, it follows that

$$\int_{-\pi}^{\pi} f(x)p_n(x)dx = 0 \text{ for all n}$$

STEP 3: **Claim:** $\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)p_n(x)dx = \infty$

This would be our desired contradiction.

Proof of Claim:

$$\int_{-\pi}^{\pi} f(x)p_n(x)dx = \left(\int_{|x|<\delta} + \int_{|x|\geq\delta}\right) f(x)p_n(x)dx$$

Study of the second term: Let $M = \sup_x |f(x)|$, then

$$\left|\int_{|x|\geq\delta} f(x)p_n(x)dx\right| \leq \int_{|x|\geq\delta} \underbrace{|f(x)|}_{\leq M} |p(x)|^n dx \leq M \int_{-\pi}^{\pi} \left(1 - \frac{\epsilon}{2}\right)^n dx$$

$$= 2\pi M \underbrace{\left(1 - \frac{\epsilon}{2}\right)^n}_{\to 0}$$

Study of the first term: Notice that p and f are ≥ 0 on $(-\delta, \delta)$ and therefore since $\eta < \delta$ we get

$$\int_{-\delta}^{\delta} f(x)p_n(x)dx \geq \int_{-\eta}^{\eta} f(x)p_n(x)dx \geq \int_{-\eta}^{\eta} \frac{f(0)}{2}\left(1 + \frac{\epsilon}{2}\right)^n dx = 2\eta\frac{f(0)}{2}\underbrace{\left(1 + \frac{\epsilon}{2}\right)^n}_{\to\infty}$$

Putting both things together we get $\int_{-\pi}^{\pi} f(x)p_n(x)dx \to \infty$ □

Corollary: If f and g are continuous on $[0, 2\pi]$ and $\hat{f}(n) = \hat{g}(n)$ for all n, then $f = g$

Consider $h = f - g$ then $\hat{h}(n) = 0$ for all n, so by the previous claim and by continuity of h, we have $h = 0$ everywhere and hence $f = g$ □

3. Uniform Convergence

Using the result above, we can finally give a positive result to our question of convergence. It's sort of a Weierstraß M test for Fourier series:

Theorem: [Uniform Convergence]

If f is a continuous 2π periodic function and moreover

$$\sum_{n=-\infty}^{\infty} \left| \hat{f}(n) \right| < \infty$$

Then the Fourier series converges uniformly to f, that is

$$\lim_{N \to \infty} S_N(f)(x) = f(x) \qquad \text{uniformly in } x$$

Proof:[3] The Fourier series of f is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

Since $\left| \hat{f}(n) e^{inx} \right| = \left| \hat{f}(n) \right| \left| e^{inx} \right| = \left| \hat{f}(n) \right|$ and $\sum_{n=-\infty}^{\infty} \left| \hat{f}(n) \right|$ converges, by the Weierstraß M-test, the above series converges uniformly to some function g, that is

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} = \lim_{N \to \infty} \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

Since the uniform limit of continuous functions is continuous, g is continuous on $[0, 2\pi]$

Moreover, if you repeat the formal calculations at the beginning (with c_n), which is now rigorous by uniform convergence, then you get that

[3] This proof is taken from Corollary 2.3 in Stein and Shakarchi

the Fourier coefficients of g are precisely $\hat{f}(n)$. But by definition $\hat{f}(n)$ are the Fourier coefficients of f. So by uniqueness we have $g = f$

In other words, the Fourier series of f converges uniformly to f ✓ $\square$

The condition on the Fourier coefficients is quite abstract. Luckily there is an easy case where this holds

4. Twice Differentiable

Theorem: If f is 2π periodic and twice continuously differentiable (f'' is continuous), then there is $C > 0$ such that for all $n \neq 0$,

$$\left|\hat{f}(n)\right| \leq \frac{C}{n^2}$$

Note: Since $\sum_{n\neq 0} \frac{1}{n^2}$ converges, the above implies that if f is twice differentiable, then the Fourier series converges to f uniformly

Proof:[4]

$$\begin{aligned}
2\pi\hat{f}(n) &= \int_{-\pi}^{\pi} f(x)e^{-inx}dx = \int_0^{2\pi} f(x)e^{-inx}dx \text{ By periodicity} \\
&\overset{\text{IBP}}{=} \left[f(x)\frac{-e^{-inx}}{in}\right]_0^{2\pi} - \int_0^{2\pi} f'(x)\frac{e^{-inx}}{-in}dx \\
&= \frac{1}{in}\int_0^{2\pi} f'(x)e^{-inx}dx \\
&= \frac{1}{in}\left[f'(x)\frac{-e^{-inx}}{in}\right]_0^{2\pi} + \frac{1}{(in)^2}\int_0^{2\pi} f''(x)e^{-inx}dx = -\frac{1}{n^2}\int_0^{2\pi} f''(x)e^{-inx}dx
\end{aligned}$$

The terms in brackets vanish since f and f' are periodic.

[4]The proof is taken from Corollary 2.4 in the Stein and Shakarchi book

Let $C = \sup_x |f''(x)|$, then

$$2\pi \left|\hat{f}(n)\right| \leq \frac{1}{n^2} \int_0^{2\pi} \underbrace{|f''(x)|}_{\leq C} \underbrace{\left|e^{-inx}\right|}_{1} dx = \frac{2\pi C}{n^2}$$

Hence $\left|\hat{f}(n)\right| \leq \frac{C}{n^2}$, as desired. □

Remark: The smoother f, the faster the decay. For example, if f is thrice differentiable, then we get $\left|\hat{f}(n)\right| \leq \frac{C}{|n|^3}$ but if f is only once differentiable, then $\left|\hat{f}(n)\right| \leq \frac{C}{|n|}$

Remark: This is only a sufficient condition. One can show that uniform convergence holds even if f is just once differentiable, even though $\sum_{n \neq 0} \frac{1}{|n|}$ diverges.