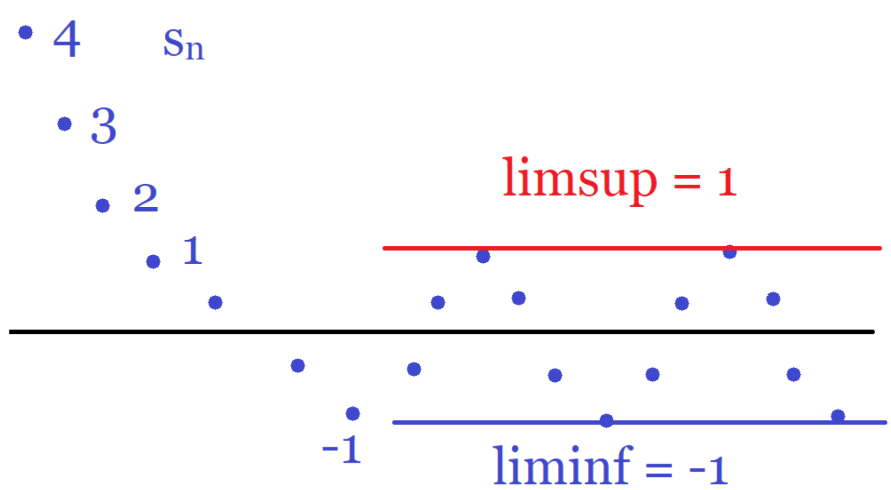
LECTURE 8: LIMSUP

1. lim sup

Video: What is lim sup?

Welcome to the second most important concept in analysis (after sup): The lim sup. Because so far we talked about convergent sequences. But in reality, lots of sequences don't converge! How do we deal with those?

Consider the following example:



Even though (s_n) doesn't converge, the *largest possible limit* (limsup) of (s_n) is 1 and the *smallest possible limit* (liminf) of (s_n) is -1.

Date: Thursday, September 23, 2021.

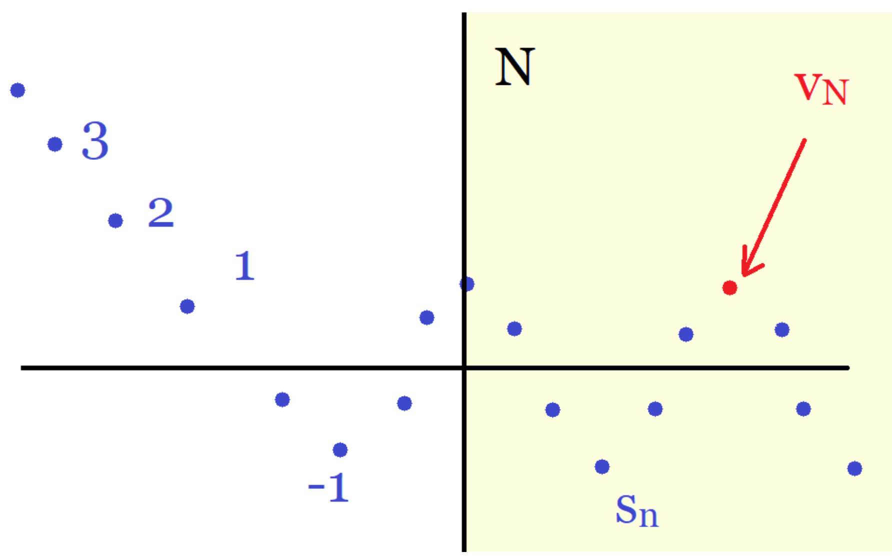
Here $\limsup$ is **NOT** the same as the sup. In this example, the sup is 4 but the limsup is 1.

Intuitively: The limsup of s_n is the sup of s_n but for *large* values of n.

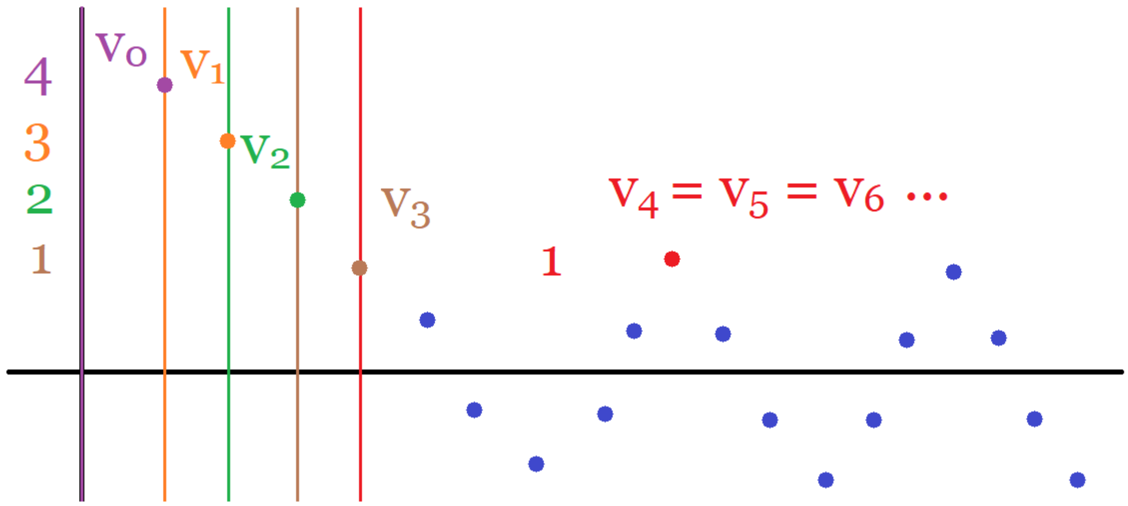
To make this more precise: Given N, define the following helper sequence (v_N) by:

$$v_N = \sup \{s_n \mid n > N\}$$

Namely, you look at the largest value of s_n, but *after* the threshold N. You ignore what's happening before N.

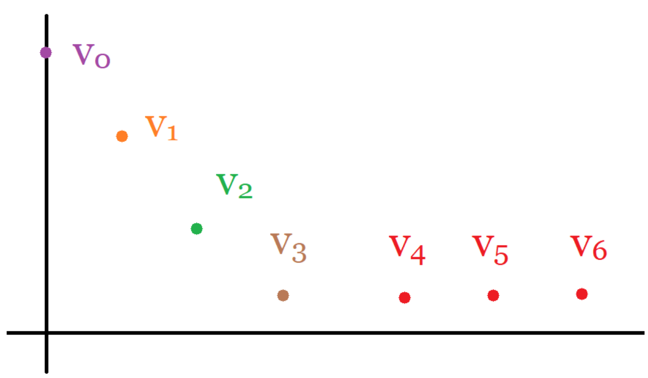


(v_N) actually has some nice properties! For this, let's plot a couple of values of v_N



$$
\begin{aligned}
v_{0} &= \sup\{s_n \mid n > 0\} = 4\\
v_1 &= \sup\{s_n \mid n > 1\} = 3\\
v_2 &= 2\\
v_3 &= 1\\
v_4 &= 1\\
v_5 &= 1
\end{aligned}
$$

Notice that the values of v_N seem to stabilize!

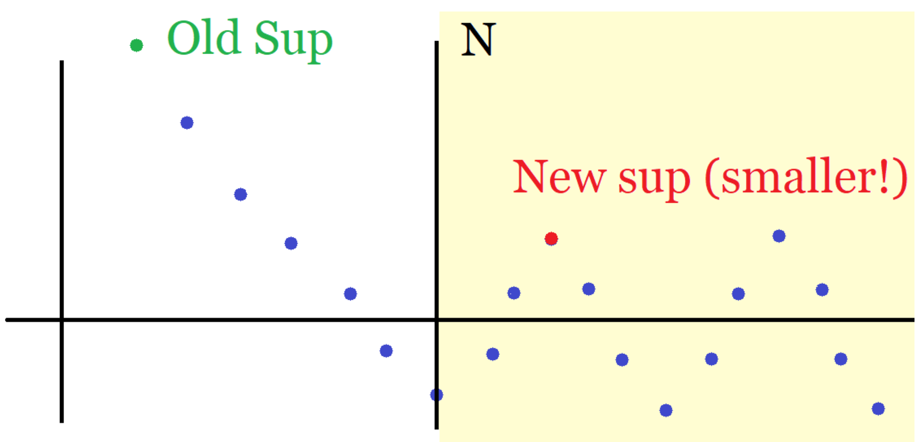


Although things generally don't always stabilize, what *is* true is that:

Fact:

(v_N) is a decreasing sequence

Why? For example, notice that for $v_0 = \sup\{s_n \mid n > 0\}$ you have *all* of values of s_n to compare. But for $v_N = \sup\{s_n \mid n > N\}$ you have much fewer choices to compare, so the sup cannot be as big as the original one!



Analogy: Suppose you have a class of 10 students and the highest score on an exam is 98. If 5 students drop, then the highest score now isn't necessarily as big any more, since some of the good students may have dropped

Since (v_N) decreasing and bounded below (if (s_n) is), by the **Monotone Sequence Theorem**, (v_N) must exist:

Fact:

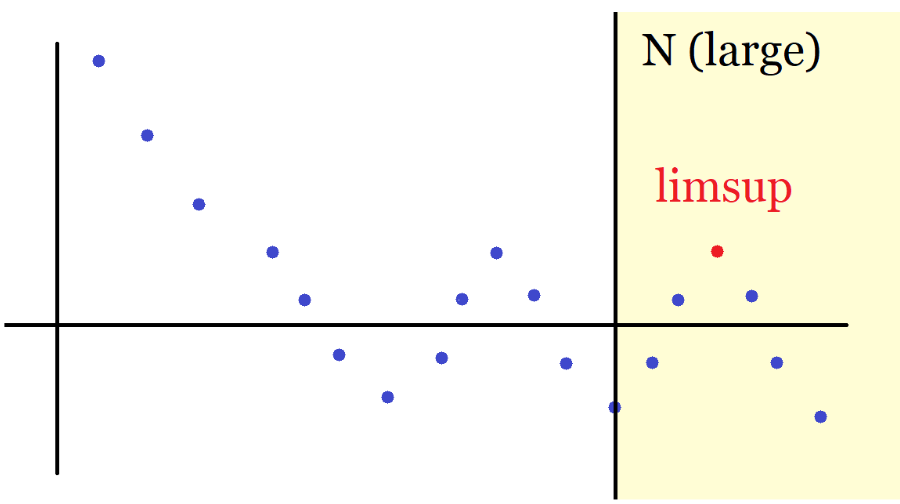
$$\lim_{N\to\infty}(v_N) \text{ exists}$$

And it is **that** limit that we call $\limsup$:

Definition:

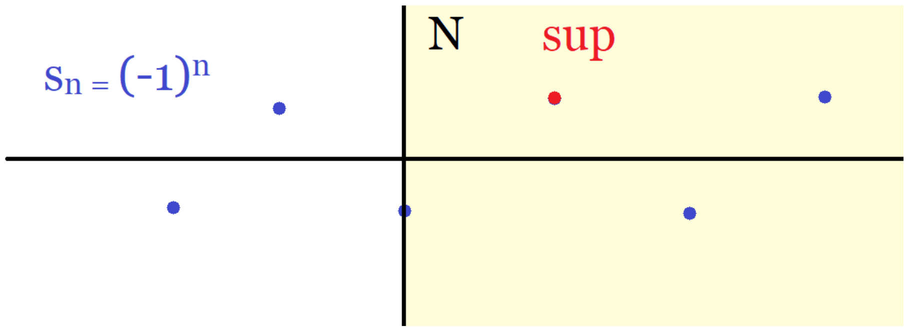
$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} v_N = \lim_{N\to\infty} \sup\{s_n \mid n > N\}$$

Interpretation: All this means is that the $\limsup$ is the sup of s_n but for *large* values of n, so it's really the *largest possible limit* of s_n



Example:

Find $\limsup_{n\to\infty} s_n$ where $s_n = (-1)^n$



Notice that for *every* N (not necessarily large),

$$v_N = \sup\{s_n \mid n > N\} = 1$$

$$\text{Hence} \quad \limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n \mid n > N\} = \lim_{N\to\infty} 1 = 1$$

Why is lim sup **SO** important? Because even though $\lim_{n\to\infty} s_n$ doesn't always exist, we have:

Upshot:

$$\limsup_{n\to\infty} s_n \text{ } \textbf{ALWAYS} \text{ exists!}$$

And in Analysis it's **GOOD** for things to exist!

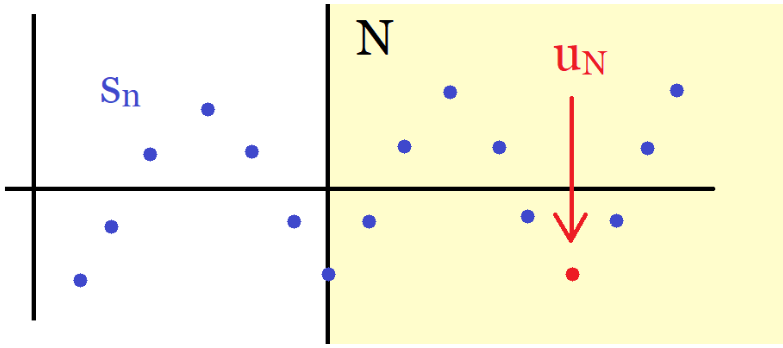
Definition:

If (s_n) is not bounded above, then

$$\limsup_{n\to\infty} s_n =: \infty$$

2. lim inf

Everything that we said for lim sup can be defined analogously with lim inf. Consider the following sequence:



This time define the helper sequence (u_N) by:

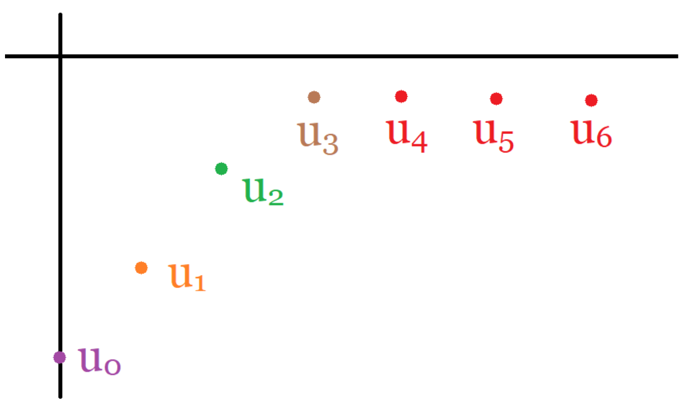
$$u_N = \inf \{s_n \mid n > N\}$$

(You look at the *smallest* value of s_n after N)

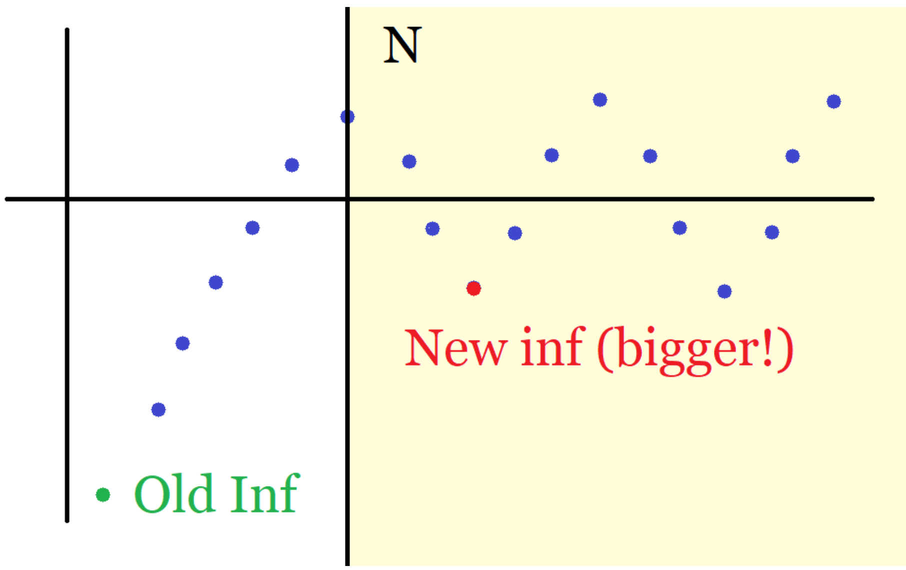
This time the opposite scenario happens, namely:

Fact:

(u_N) is an *increasing* sequence



Why? If $N = 0$, then we have all the values of s_n to compare, so the inf can be really small. But as N gets bigger, we have fewer and fewer values to compare, so the inf can't be that small any more



Analogy: If you have 10 students and the lowest score is 20%. Now suppose 5 (bad) students dropped. Then the lowest score is now (probably) higher.

And since (u_N) is increasing and bounded above, by the Monotone Sequence Theorem, (u_N) converges, and **that** limit is called lim inf

Definition:

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} u_N = \lim_{N\to\infty} \inf \{s_n \mid n > N\}$$

Example: (more practice)

Find $\liminf_{n\to\infty} s_n$ where $s_n = (-1)^n$

Notice that for *every* N (not necessarily large),

$$u_N = \inf \{s_n \mid n > N\} = -1$$

$$\text{Hence} \quad \liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf \{s_n \mid n > N\} = \lim_{N\to\infty} -1 = -1$$

And just as before:

Upshot:

$$\liminf_{n\to\infty} s_n \textbf{ ALWAYS } \text{exists!}$$

Definition:

If (s_n) is not bounded below, then we define

$$\liminf_{n\to\infty} s_n = -\infty$$

3. $\liminf$ VS $\limsup$

The good news is that we never have to deal with $\liminf$ explicitly, because we have the following identity:

Fact:

$$\liminf_{n\to\infty} s_n = -\limsup_{n\to\infty}(-s_n)$$

Why? Recall that for any set S we have:

$$\inf(S) = -\sup(-S)$$

Use the above identity with $S = \{s_n \mid n > N\}$, then $-S = \{-s_n \mid n > N\}$ and the above identity becomes

$$\inf \{s_n \mid n > N\} = -\sup \{-s_n \mid n > N\}$$

Finally take $\lim_{N\to\infty}$ on both sides:

$$\lim_{N\to\infty} \inf \{s_n \mid n > N\} = \lim_{N\to\infty} -\sup \{-s_n \mid n > N\}$$
$$\lim_{N\to\infty} \inf \{s_n \mid n > N\} = -\lim_{N\to\infty} \sup \{-s_n \mid n > N\}$$
$$\liminf_{n\to\infty} s_n = -\limsup_{n\to\infty} (-s_n) \quad \square$$

Next, we'll discuss two important theorems related to lim sup:

4. lim sup and Convergence

Video: Limsup vs Convergence

The *amazing* fact is that even if (s_n) doesn't always converge, the limsup always exists. But what if s_n converges to s? Then the lim sup must be equal to s:

Theorem:

If (s_n) converges to s, then:

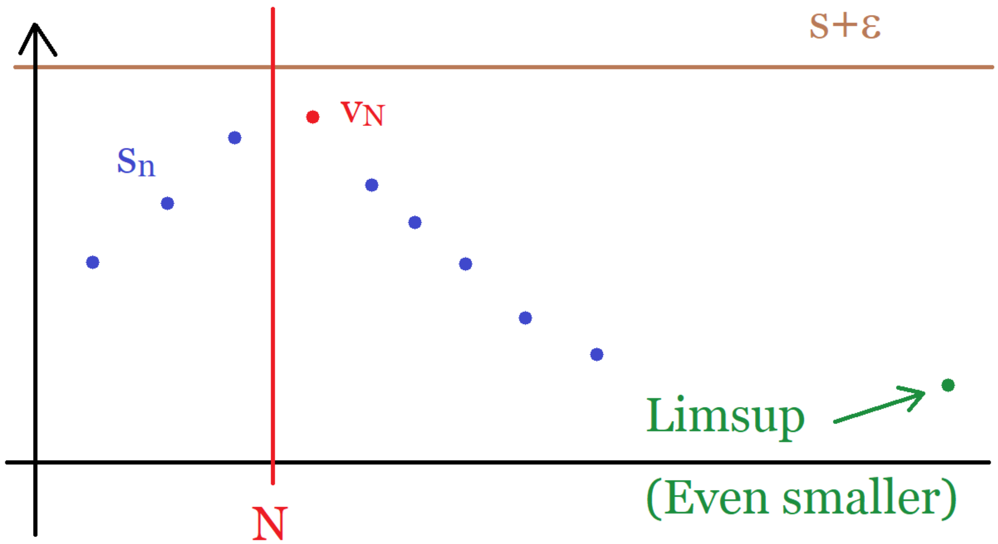
$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$$

Proof:

STEP 1: Let $\epsilon > 0$ be given, then there is N such that if $n > N$, then $|s_n - s| < \epsilon$. But $|s_n - s| < \epsilon \Rightarrow s_n - s < \epsilon \Rightarrow s_n < s + \epsilon$. But since this is true for *all* $n > N$, we get

$$\sup \{s_n \mid n > N\} \leq s + \epsilon \Rightarrow v_N \leq s + \epsilon$$

(Remember (v_N) was our helper sequence, it's the sup after N)



⚠ Here N is **fixed**, we cannot change it

But since (v_N) is decreasing, and the $\limsup_{n\to\infty} s_n$ is even smaller, and so

$$\limsup_{n\to\infty} s_n \leq v_N \leq s + \epsilon \Rightarrow \limsup_{n\to\infty} s_n \leq s + \epsilon$$

Since ϵ was arbitrary we get $\limsup_{n\to\infty} s_n \leq s$

STEP 2: First of all, notice $-s_n \to -s$, so using the result from **STEP 1**, we get $\limsup_{n\to\infty} -s_n \leq -s$. Then, by the identity relating $\liminf$ and $\limsup$, we get:

$$\liminf_{n\to\infty} s_n = -\limsup_{n\to\infty} -s_n \geq -(-s) = s$$

Hence $\liminf_{n\to\infty} s_n \geq s$

STEP 3: Finally, using **STEP 2**, $\liminf \leq \limsup$, and **STEP 1**, we obtain:

$$s \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq s$$

$$\text{Hence} \quad \liminf_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s \quad \square$$

5. lim sup Squeeze Theorem

Video: lim sup Squeeze Theorem

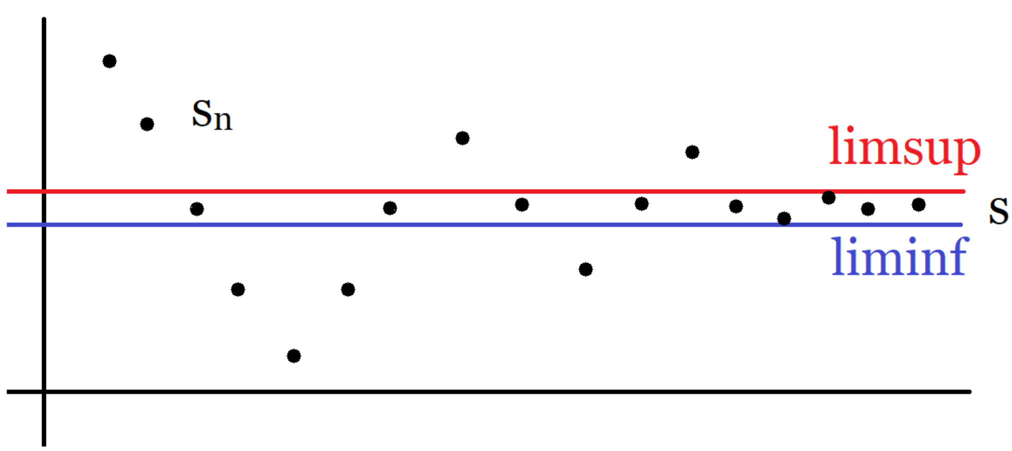
Not only that, you can use lim inf and lim sup to actually show that a sequence converges! It's like a squeeze theorem for lim inf and lim sup:

lim sup Squeeze Theorem:

Suppose

$$\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = s$$

Then (s_n) converges to s



Intuitively, this theorem makes sense: lim sup is the largest possible limit of (s_n) and lim inf is the smallest possible limit, so if both of them agree, this forces (s_n) to converge to s, kind of like the squeeze theorem.

Proof:

STEP 1: Let $\epsilon > 0$ be given.

By assumption, we know $\limsup_{n\to\infty} s_n = s$, that is:

$$\lim_{N\to\infty} v_N = s$$

Therefore, by definition of the limit, there is N_1 such that if $N > N_1$, then

$$\begin{aligned} |v_N - s| &< \epsilon \\ -\epsilon < v_N - s &< \epsilon \\ v_N - s &< \epsilon \\ v_N &< s + \epsilon \\ \sup\{s_n \mid n > N\} &< s + \epsilon \end{aligned}$$

But by definition of a sup, this means that for *all* $n > N$ we have $s_n < s + \epsilon \Rightarrow s_n - s < \epsilon$.

STEP 2: Similarly, since $\liminf_{n\to\infty} s_n = s$, there is N_2 such that if $N > N_2$, then

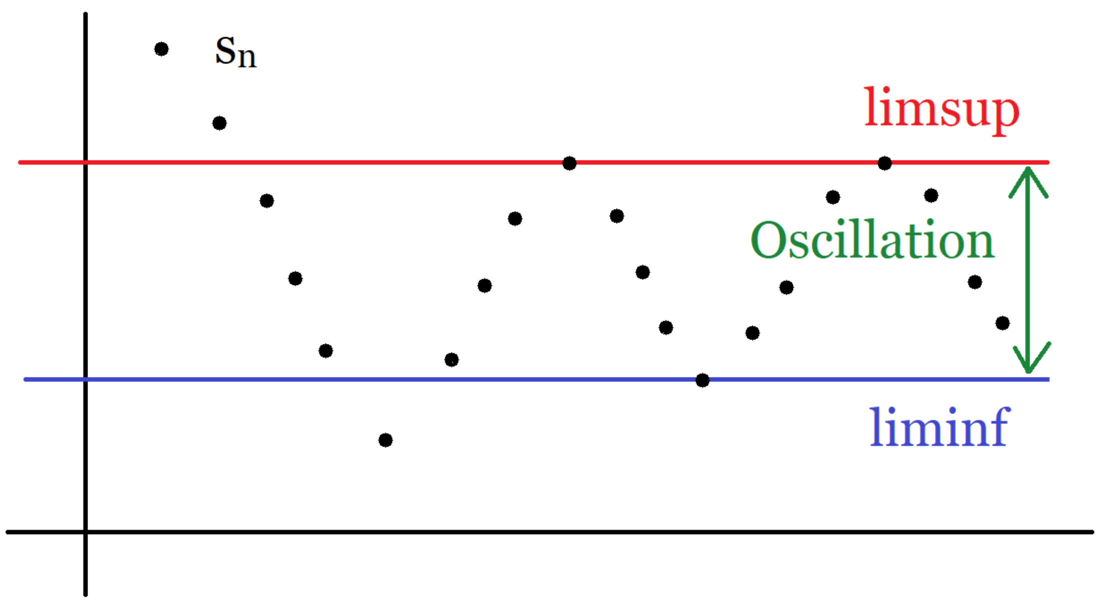
$$\begin{aligned} |u_N - s| &< \epsilon \\ -\epsilon < u_N - s &< \epsilon \\ u_N - s &> -\epsilon \\ u_N &> s - \epsilon \\ \inf\{s_n \mid n > N\} &> s - \epsilon \end{aligned}$$

But by definition of inf, this means that for all $n > N$ we have $s_n > s - \epsilon \Rightarrow s_n - s > -\epsilon$.

STEP 3: Given $\epsilon > 0$, let $N = \max\{N_1, N_2\}$ as above, then if $n > N$ both conditions in **STEP 1** and **STEP 2** hold, so we have $s_n - s < \epsilon$ and $s_n - s > -\epsilon$, that is $|s_n - s| < \epsilon$. ✓

Hence (s_n) converges to s □

Note: This theorem implies that if (s_n) doesn't converge, then $\limsup > \liminf$, meaning (s_n) has positive *oscillation*.

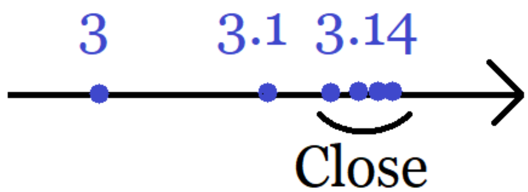


6. Cauchy Sequences

Video: Cauchy Sequences

Finally, let's discuss an **important** topic related to convergence.

Motivation: Consider $(s_n) = (3, 3.1, 3.14, 3.141, \dots)$. Notice that the terms of s_n get closer and closer to each other

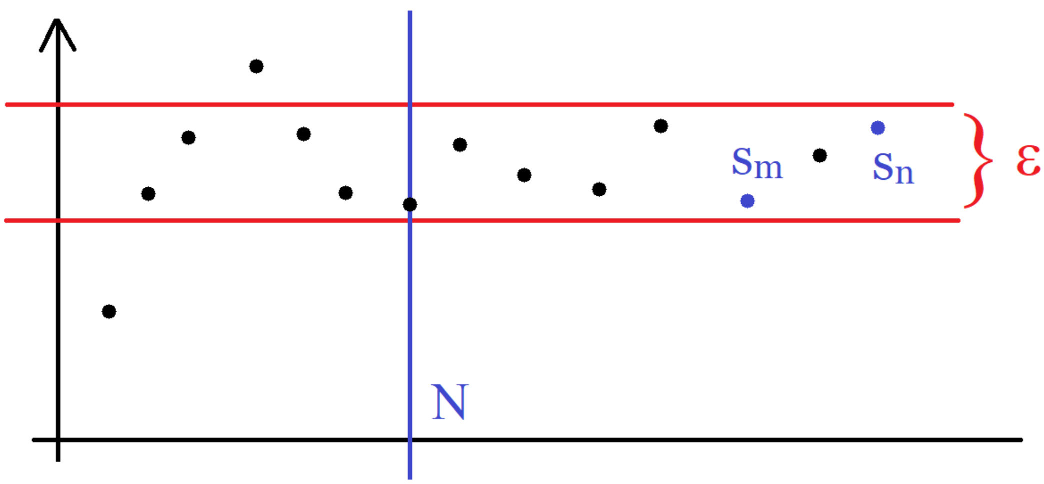


It's precisely this phenomenon that we call a *Cauchy sequence:*

Definition:

We say (s_n) is a **Cauchy sequence** if for all $\epsilon > 0$ there is N such that if $m, n > N$, then

$$|s_n - s_m| < \epsilon$$



In other words, in the long run, all the terms of the sequence become close to each other.

Important: This is **NOT** the same as convergence, which says that for all $\epsilon > 0$ there is N such that if $n > N$, then $|s_n - s| < \epsilon$. There is a subtle difference here: Convergence means that all the terms are eventually close to **a fixed number** s, whereas Cauchy means that all the terms are eventually close to **each other**.

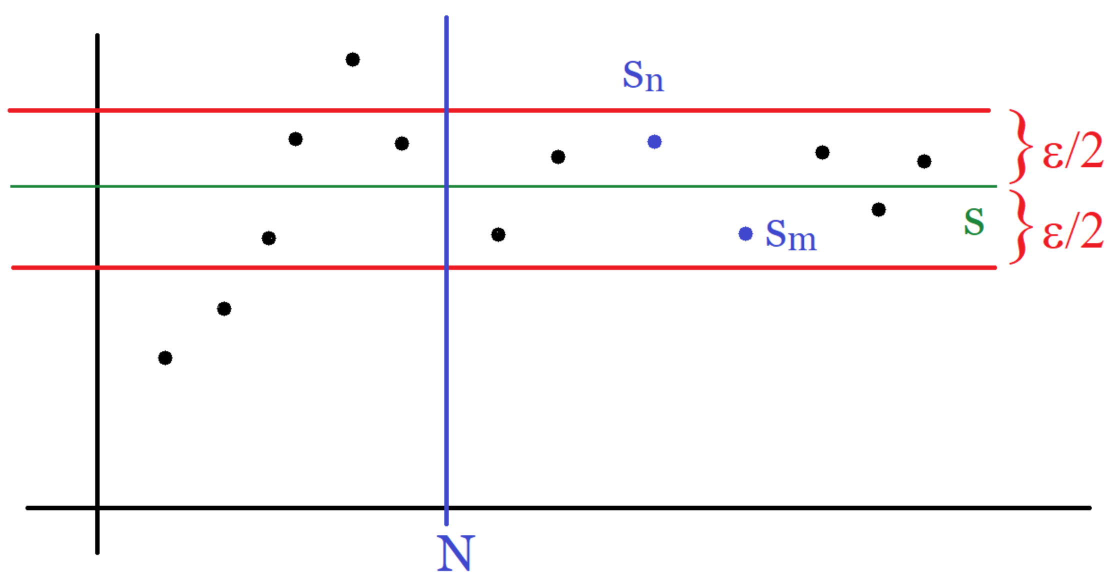
Notice that in the definition of Cauchy, there is no mention of the limit whatsoever. In fact, we'll see later that not all Cauchy sequences are convergent. However, what *is* true is that all convergent sequences are Cauchy:

Convergence $\Rightarrow$ Cauchy

If (s_n) converges to s, then (s_n) is Cauchy

Analogy: If everyone is going to a concert hall (= (s_n) converges), then the concert hall will be crowded (= (s_n) is Cauchy).

Proof: Let $\epsilon > 0$ be given.



Then there is N such that if $n > N$, then $|s_n - s| < \frac{\epsilon}{2}$.

But this also means that if $m > N$, then $|s_m - s| < \frac{\epsilon}{2}$ as well (we're just renaming the variables: If it's true for n it's true for m as well)

Therefore, if m and n are $> N$, then

$$|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n| = |s_m - s| + |s_n - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \checkmark$$

Hence (s_n) is Cauchy $\square$

What about the converse? Does Cauchy $\Rightarrow$ Convergence? This is a very interesting question, and will be answered next time!