LECTURE 8: VECTOR FUNCTIONS (III)

Finally, let's conclude our exploration of vector functions with some nice physical applications

1. Velocity and Acceleration

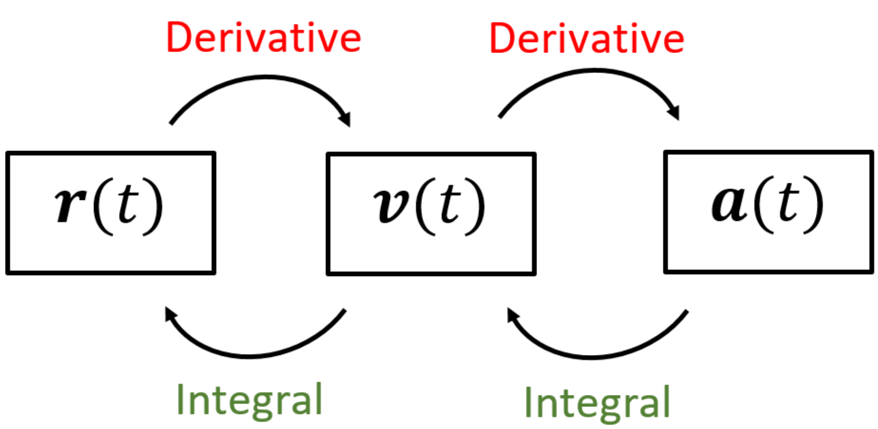
Just like in single-variable calculus, where $f(x)$ represents position, $f'(x)$ velocity, and $f''(x)$ acceleration, we have the same interpretation for vector functions:

Example 1:

Suppose the position of a particle is given by

$$\mathbf{r}(t) = \left\langle 1 + 3t^2, 4t, 2t^3 \right\rangle$$

Find the velocity $\mathbf{v}(t)$, the speed $\|\mathbf{v}(t)\|$, and the acceleration $\mathbf{a}(t)$



Date: Wednesday, September 15, 2021.

Definition:

$$\begin{aligned} \text{Velocity } &= \mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 6t, 4, 6t^2 \right\rangle \\ \text{Speed } &= \|\mathbf{v}(t)\| = \left\| \left\langle 6t, 4, 6t^2 \right\rangle \right\| = \sqrt{36t^2 + 16 + 36t^4} \\ \text{Acceleration } &= \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle 6, 0, 12t \rangle \end{aligned}$$

(Fun fact, the derivative of acceleration is the jerk, then it's snap, crackle, pop)

Example 2:

Suppose the position of a particle is given by

$$\mathbf{r}(t) = \left\langle e^{2t}, \cos(t), \sin(t) \right\rangle$$

Find the velocity, speed, and acceleration at time $t = \pi$

Velocity:

$$\begin{aligned} \mathbf{v}(t) =& \mathbf{r}'(t) = \left\langle 2e^{2t}, -\sin(t), \cos(t) \right\rangle \\ \mathbf{v}(\pi) =& \left\langle 2e^{2\pi}, -\sin(\pi), \cos(\pi) \right\rangle = \left\langle 2e^{2\pi}, 0, -1 \right\rangle \end{aligned}$$

Speed:

$$\|\mathbf{v}(\pi)\| = \sqrt{\left(2e^{2\pi}\right)^2 + 0^2 + (-1)^2} = \sqrt{4e^{4\pi} + 1}$$

Acceleration:

$$\begin{aligned} \mathbf{a}(t) =& \mathbf{v}'(t) = \left\langle 4e^{2t}, -\cos(t), -\sin(t) \right\rangle \\ \mathbf{a}(\pi) =& \left\langle 4e^{2\pi}, -\cos(\pi), \sin(\pi) \right\rangle = \left\langle 4e^{2\pi}, 1, 0 \right\rangle \end{aligned}$$

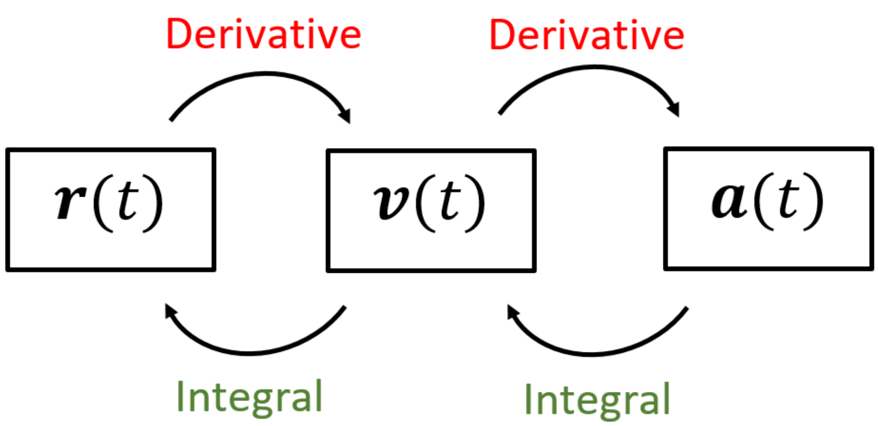
In practice we need to do this in reverse: Given an acceleration, figure out the velocity and position.

Example 3: (Good Quiz/Exam Question)

Suppose the acceleration of a particle is given by

$$\mathbf{a}(t) = \langle 0, 2, 3 \rangle$$

Find the position $\mathbf{r}(t)$ given that initially, the particle is at $\langle 1, 0, -1 \rangle$ and moves with velocity $\langle 1, 1, 2 \rangle$.



Since the derivative of velocity is acceleration, we get that:

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)dt \\ &= \left\langle \int 0dt, \int 2dt, \int 3dt \right\rangle \\ &= \langle A, 2t + B, 3t + C \rangle\end{aligned}$$

To figure out A, B, C, plug in $t = 0$

$\mathbf{v}(0) = \langle A, 2(0) + B, 3(0) + C \rangle = \langle A, B, C \rangle = \langle 1, 1, 2 \rangle$ (from the question)

Hence $A = 1, B = 1, C = 2$ and we get

$$\mathbf{v}(t) = \langle 1, 2t + 1, 3t + 2 \rangle$$

And since $\mathbf{v}(t) = \mathbf{r}'(t)$ we get:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)dt \\ &= \left\langle \int 1dt, \int 2t + 1dt, \int 3t + 2dt \right\rangle \\ &= \left\langle t + D, t^2 + t + E, \frac{3}{2}t^2 + 2t + F \right\rangle\end{aligned}$$

And again to figure out D, E, F, plug in $t = 0$:

$$\mathbf{r}(0) = \left\langle 0 + D, 0^2 + 0 + E, \frac{3}{2}0^2 + 2(0) + F \right\rangle = \langle D, E, F \rangle = \langle 1, 0, -1 \rangle$$

Hence $D = 1, E = 0, F = -1$ and therefore

$$\mathbf{r}(t) = \left\langle t + 1, t^2 + t, \frac{3}{2}t^2 + 2t - 1 \right\rangle$$

2. May the Force be with you

Using this and Newton's Second Law of Motion, you can also find the force acting on an object:

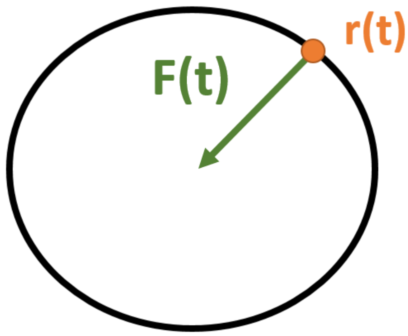
Example 4: (Force)

Suppose a particle of mass $m = 4$ has position

$$\mathbf{r}(t) = \langle 3\cos(2t), 3\sin(2t) \rangle$$

Find the force $\mathbf{F}(t)$ acting on the particle, as well as its magnitude $\|\mathbf{F}(t)\|$

Physical Interpretation: Here the particle is moving on a circle of radius 3 with "angular speed" 2, and we need to find the force acting on it.



First, find the acceleration:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 3(-2\sin(2t)), 3(2\cos(2t))\rangle = \langle -6\sin(2t), 6\cos(2t)\rangle$$
$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -6(2\cos(2t)), 6(-2\sin(2t))\rangle = \langle -12\cos(2t), -12\sin(2t)\rangle$$

To find the Force, use:

Newton's Second Law (Memorize)

$$\mathbf{F} = m\,\mathbf{a}$$

Here with $m = 4$ we get

$$\mathbf{F}(\mathbf{t}) = 4\mathbf{a}(t) = 4\,\langle -12\cos(2t), -12\sin(2t)\rangle = \langle -48\cos(2t), -48\sin(2t)\rangle$$

And also

$$
\begin{aligned}
\|\mathbf{F(t)}\| &= \|\langle -48\cos(2t), -48\sin(2t)\rangle\| \\
&= |-48|\,\|\langle \cos(2t), \sin(2t)\rangle\| \\
&= 48\sqrt{\cos^2(2t) + \sin^2(2t)} \\
&= 48
\end{aligned}
$$

(Beware of the absolute value, since here the force is positive)

Interpretation: Here the force points towards the origin (like in the picture above), which is sometimes called a *centripetal force*.

Note: At the end of the notes, there's another (optional) physics application, which deals with rockets and projectiles.

3. 8 Important Surfaces (section 12.6)

This concludes our vector function adventure, and we are finally ready for the main protagonists of this course, eight important surfaces that you'll see over and over again.

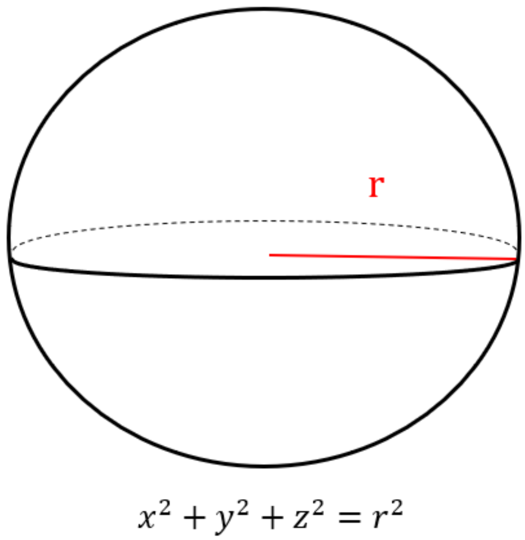
Important: Please memorize those surfaces, they're extremely important!

3.1. Spheres and Ellipsoids.

Surface 1: Sphere

$$x^2 + y^2 + z^2 = r^2$$

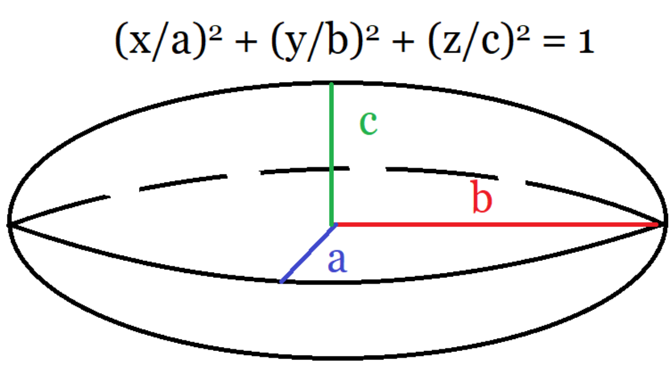
This is a typical sphere of radius r



Surface 2: Ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Just a compressed version of a sphere, which is an ellipsoid (3D version of ellipse)



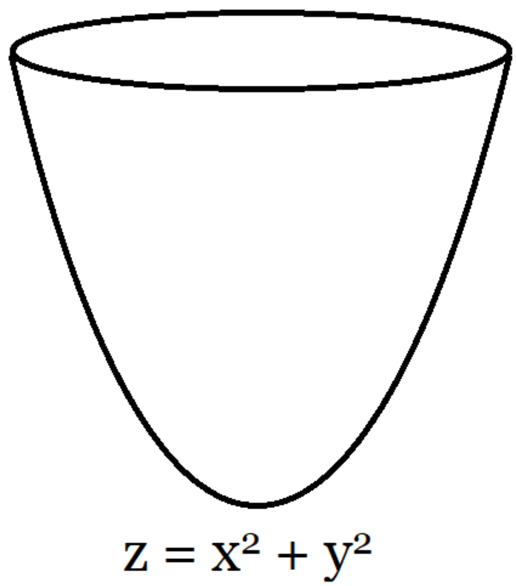
Note: Also applies to surfaces like $2x^2 + 3y^2 + 7z^2 = 1$

3.2. $z =$ Something .

Surface 3: (Elliptic) Paraboloid

$$z = x^2 + y^2$$

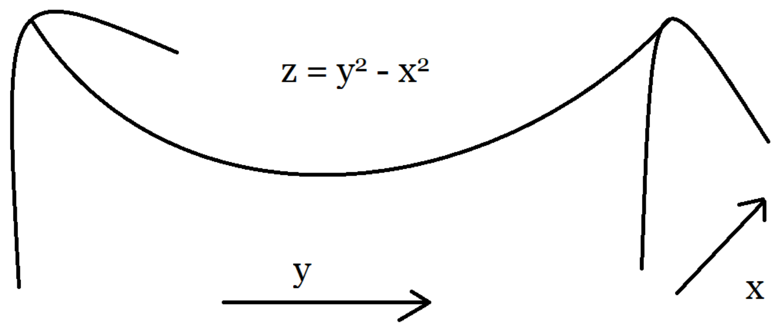
Think a generalization of $y = x^2$, like a $3D$ parabola.



Surface 4: Saddle / (Hyperbolic) Paraboloid

$$z = y^2 - x^2$$

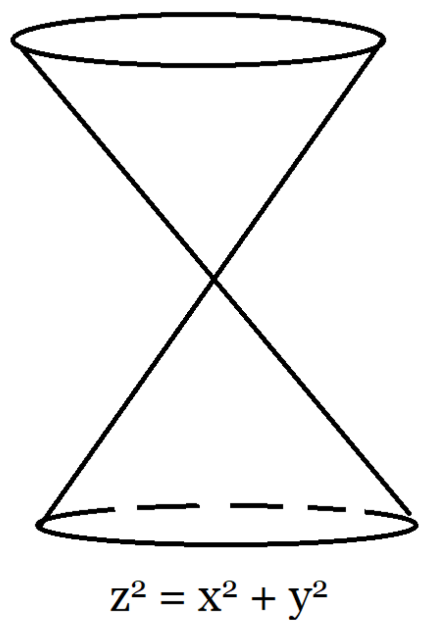
Notice that it's increasing in y (because of y^2) but decreasing in x (because of $-x^2$), so it looks like a horse saddle:



3.3. **Cone.**

Surface 5: Cone

$$z^2 = x^2 + y^2$$

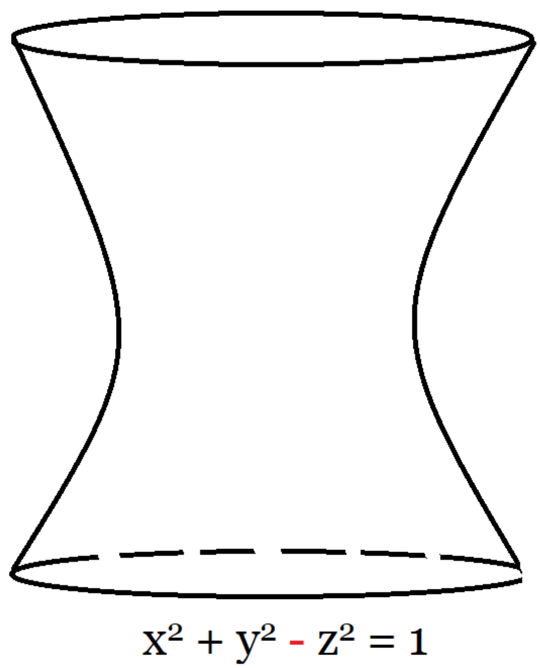


(Think circles that are getting bigger in both directions)

3.4. **One or two minuses.**

Surface 6: Dress / Hyperboloid of One Sheet

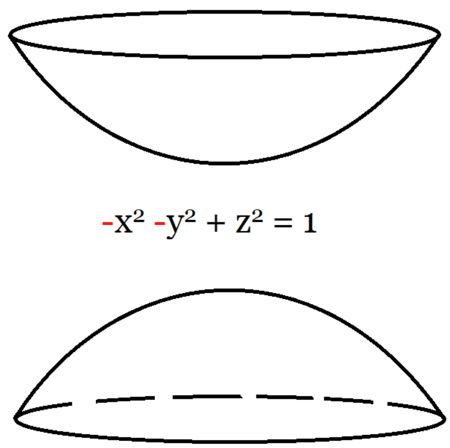
$$x^2 + y^2 - z^2 = 1$$



One minus, so hyperboloid of **one** sheet. Looks like a dress or nuclear reactor. It's important that the 1 above is a 1 (or any positive number).

Surface 7: Two Cups / Hyperboloid of Two Sheets

$$-x^2 - y^2 + z^2 = 1$$

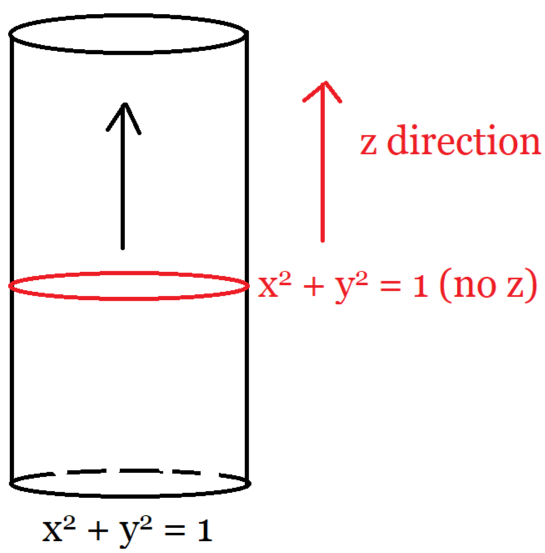


Two minuses, so hyperboloid of **two** sheets. Looks like two cups.

3.5. Cylinder.

Surface 8: Cylinder

$$x^2 + y^2 = 1$$



Whenever a variable is missing, it's a cylinder, in the direction of the missing variable.

Note: You will need to be able to draw/recognize the surfaces in different directions (like $x^2 = y^2 + z^2$, which is a cone in the $x-$direction) and also be able to complete the square, like the sphere example in the first lecture. Here's a fun video illustrating this:

Video: Kamehameha Equation

4. To the moon (optional)

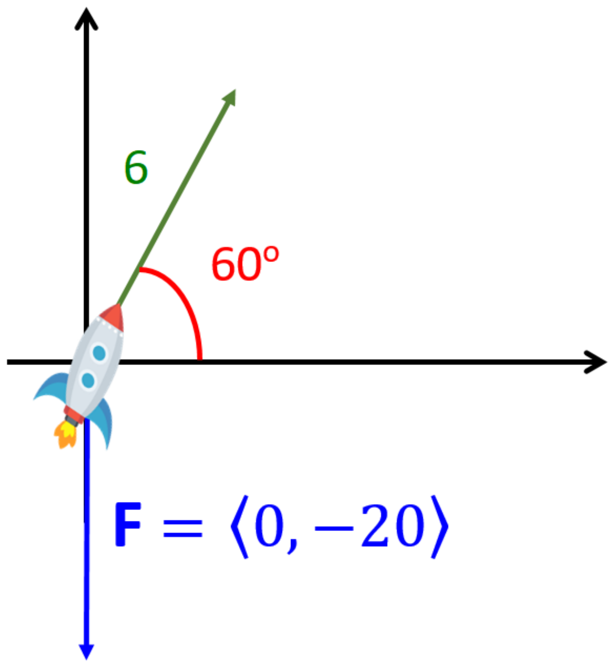
Here is another neat application of vector functions to projectiles.

Example 5: (Projectile)

Suppose a projectile with mass $m = 2$ is fired with angle of elevation 60° and initial speed 6.

Find the position $\mathbf{r}(t)$ of the projectile.

Assume no friction, and the only force is gravity. Use $g = 10m/s^2$



STEP 1: Suppose (for simplicity) that the projectile starts at the origin $(0, 0)$. Since the force due to gravity is vertical and downward, we have:

$$\mathbf{F} = -mg\,\langle 0, 1\rangle = -2(10)\,\langle 0, 1\rangle = \langle 0, -20\rangle$$

Now by Newton's Second Law, we have

$$\mathbf{F} = m\mathbf{a} = 2\mathbf{a}(t)$$

Comparing the two, we get

$$2\mathbf{a}(t) = \langle 0, -20\rangle \Rightarrow \mathbf{a}(t) = \langle 0, -10\rangle$$

STEP 2: Therefore the velocity becomes

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt = \left\langle \int 0dt, \int -10dt \right\rangle = \langle A, -10t + B \rangle = \langle 0, -10t \rangle + \langle A, B \rangle$$

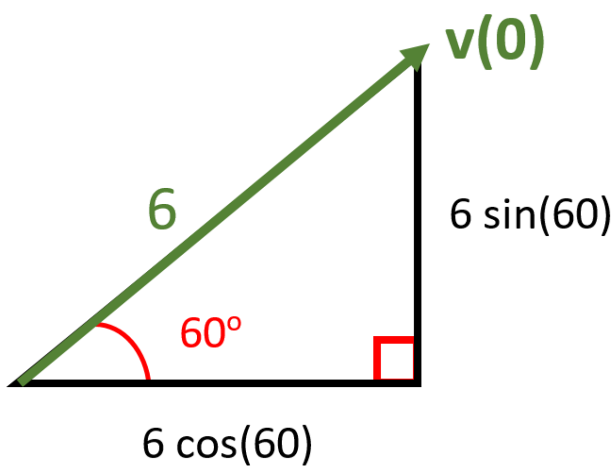
STEP 3: Study of $\mathbf{v}(0)$

Plugging $t = 0$ in the above, we have:

$$\mathbf{v}(0) = \langle 0, -10(0) \rangle + \langle A, B \rangle = \langle A, B \rangle$$

So $\langle A, B \rangle$ is nothing other than the initial velocity $\mathbf{v}(0)$.

By definition, the initial speed is $\|\mathbf{v}(\mathbf{0})\| = 6$, and the angle is $60°$. This allows us to find $\mathbf{v}(0)$, as in the picture below:



By SOHCAHTOA (think polar coordinates) we get:

$$\mathbf{v}(0) = \langle 6\cos(60°), 6\sin(60°) \rangle = \left\langle 6\left(\frac{1}{2}\right), 6\left(\frac{\sqrt{3}}{2}\right) \right\rangle = \left\langle 3, 3\sqrt{3} \right\rangle$$

And therefore, from **STEP 2**, we have:

$$\mathbf{v}(t) = \langle 0, -10t\rangle + \underbrace{\langle A, B\rangle}_{\mathbf{v(0)}} = \langle 0, -10t\rangle + \left\langle 3, 3\sqrt{3}\right\rangle = \left\langle 3, 3\sqrt{3} - 10t\right\rangle$$

STEP 4: Finally,

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t) \\ &= \left\langle \int 3dt, \int 3\sqrt{3} - 10tdt \right\rangle \\ &= \left\langle 3t + C, 3\sqrt{3}t - 5t^2 + D\right\rangle\end{aligned}$$

But since the projectile is initially at $(0,0)$, we have $\mathbf{r}(0) = \langle 0, 0\rangle$, and

$$\left\langle 3(0) + C, 3\sqrt{3}(0) - 5(0)^2 + D\right\rangle = \langle C, D\rangle = \langle 0, 0\rangle$$

And so $C = 0$ and $D = 0$, and therefore

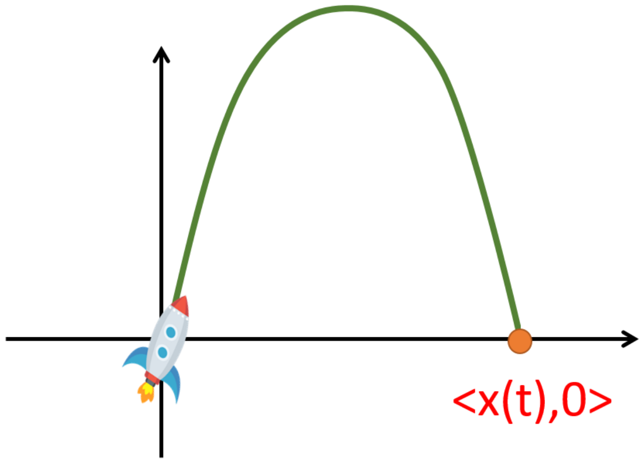
$$\mathbf{r}(t) = \left\langle 3t, 3\sqrt{3}t - 5t^2\right\rangle$$

(Notice the trajectory is actually a parabola)

Example 6:

(a) At which time and point does the projectile from the previous problem hit the ground, and at what speed?

(b) At which time does the projectile have maximum height?



(a) **Time:** All we need to know is when $y(t) = 0$, that is:

$$\begin{aligned}3\sqrt{3}t - 5t^2 =& 0\\ t(3\sqrt{3} - 5t) =& 0\\ 3\sqrt{3} - 5t =& 0 \quad \text{(ignoring } t = 0)\\ 5t =& 3\sqrt{3}\\ t =& \frac{3\sqrt{3}}{5} \approx 1.03\end{aligned}$$

Point: At this time we have $y(t) = 0$ and

$$x(t) = 3t = 3\left(\frac{3\sqrt{3}}{5}\right) = \frac{9\sqrt{3}}{5} \approx 3.09$$

So the point is $\left(\frac{9\sqrt{3}}{5}, 0\right) \approx (3.09, 0)$.

Speed: Finally, using $\mathbf{v}(t) = \left\langle 3, 3\sqrt{3} - 10t \right\rangle$, at that time, we have

$$3\sqrt{3} - 10t = 3\sqrt{3} - 10\left(\frac{3\sqrt{3}}{5}\right) = 3\sqrt{3} - 6\sqrt{3} = -3\sqrt{3}$$

So at that time $\mathbf{v}(t) = \left\langle 3, -3\sqrt{3} \right\rangle$ and the speed is

$$\|\mathbf{v}(t)\| = \sqrt{3^2 + (-3\sqrt{3})^2} = \sqrt{9 + 27} = \sqrt{36} = 6$$

(b) Here we just need to maximize $y(t) = 3\sqrt{3}t - 5t^2$:

$$y'(t) = 3\sqrt{3} - 10t = 0 \Rightarrow t = \frac{3\sqrt{3}}{10}$$

Also $y''(t) = -10 < 0$

So y indeed has a maximum at $t = \frac{3\sqrt{3}}{10} \approx 0.52$