LECTURE 8: CONVOLUTION

1. Dirichlet Kernel

One quantity of importance in the study of Fourier series is the Dirichlet kernel, defined as follows:

Definition: [Dirichlet Kernel]

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

(Think of it as a function with Fourier coefficients 1 and then 0)

The cool thing is that there is actually a closed formula for D_N

Fact:

$$D_N(x) = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}$$

Why? Can do it directly, using geometric sums[1], or consider the following:

$$\left(e^{ix}-1\right) D_N(x) = e^{ix} D_N - D_N = \sum_{n=-N}^{N} e^{i(n+1)x} - \sum_{n=-N}^{N} e^{inx} = e^{i(N+1)x} - e^{-iNx}$$

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[1] See Example 4 in Chapter 2 of Stein and Shakarchi

The last step is because we have a telescoping sum, so the only thing that remains is the last $(N+1)$ term and the first $(-N)$ term.

Multiplying both sides by $e^{\frac{-ix}{2}}$ and dividing by $2i$ we get

$$\frac{e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}}{2i} D_N(x) = \frac{e^{i\left(N+\frac{1}{2}\right)x} - e^{-i\left(N+\frac{1}{2}\right)x}}{2i}$$

And since $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$ we get

$$\sin\left(\frac{x}{2}\right) D_N(x) = \sin\left(\left(N+\frac{1}{2}\right)x\right) \quad \square$$

2. Relation to Fourier Series

Recall:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \text{ Fourier coefficient}$$

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \text{ Partial Sums}$$

It turns out that we can conveniently write $S_N(f)$ in terms of D_N

Important Observation:

$$\begin{aligned}
S_N(f)(x) &= \sum_{n=-N}^{N} \hat{f}(n) e^{inx} \\
&= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy \right) e^{inx} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^{N} e^{in(x-y)} \right) dy \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_N(x-y) dy
\end{aligned}$$

We have therefore shown that:

Fact:

$$S_N(f)(x) = (f \star D_N)(x)$$

Where $\star$ is an operation called **convolution** that we'll talk about below.

In other words, the problem of understanding $S_N(f)$ reduces to the understanding of the convolution $f \star D_N$. Here is a great illustration of this fact:

3. Pointwise Convergence

Let's show that if f is (sort of) Lipschitz, then the Fourier series of f converges to f pointwise.

Definition: f is **Lipschitz at** x there is $L > 0$ such that for all y,

$$|f(y) - f(x)| \leq L\,|y - x|$$

Here L can depend on x.

Theorem [Pointwise Convergence]

Suppose f is Lipschitz at x, then

$$\lim_{N\to\infty} S_N(f)(x) = f(x)$$

Note: A similar proof works if f is differentiable at x, see Stein and Shakarchi Theorem 2.1.

Corollary: If f is Lipschitz or f is differentiable, then the Fourier series converges pointwise everywhere.

Proof:[2]

STEP 1: Notice from the def of $D_N = \sum_n e^{inx}$ that $\int_{-\pi}^{\pi} D_N(y) = 2\pi$ and therefore:

$$\begin{aligned}
S_N(f)(x) - f(x) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)D_N(y)dy - f(x)\frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi} D_N(y)dy}_{2\pi} \\
=& \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x-y) - f(x)\right) D_N(y)dy \\
=& \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x-y) - f(x)\right) \left(\frac{\sin\left(\left(N+\frac{1}{2}\right)y\right)}{\sin\left(\frac{y}{2}\right)}\right) dy
\end{aligned}$$

In the last step we used the explicit formula for D_N above.

[2]The proof is adapted from Theorem 8.14 in Rudin

STEP 2: Using that

$$\sin\left(\left(N+\frac{1}{2}\right)y\right)=\sin(Ny)\cos\left(\frac{y}{2}\right)+\cos(Ny)\sin\left(\frac{y}{2}\right)$$

The above becomes

$$\begin{aligned}S_N(f)(x)-f(x)=&\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-y)-f(x))\left(\frac{\cos\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)}\right)\sin(Ny)dy\\&+\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-y)-f(x))\cos(Ny)dy\end{aligned}$$

Upshot: Notice that those two integrals kind of look like Fourier coefficients!

STEP 3: To conclude the proof, we will need a result that we'll prove next time:

Fact: If g is bounded and integrable on $[-\pi,\pi]$, then

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}g(y)\cos(Ny)dy=0 \text{ and } \lim_{N\to\infty}\int_{-\pi}^{\pi}g(y)\sin(Ny)=0$$

Our proof then follows, once we show that the functions are bounded and integrable.

For the second term, boundedness follows from the Lipschitz condition:

$$|f(x-y)-f(x)|\leq L|y|\to \text{ bounded on } [-\pi,\pi]$$

And by continuity of f, the function above is integrable on $[-\pi,\pi]$.

$$\left|(f(x-y)-f(x))\cot\left(\frac{y}{2}\right)\right|\leq L|y|\left|\cot\left(\frac{y}{2}\right)\right|\to \text{ bounded on } [-\pi,\pi]$$

For boundedness, the only point of concern is near $y = 0$. But it is not a problem since $\lim_{y\to 0} y \cot\left(\frac{y}{2}\right) = 2$ (by L'Hôpital), and integrability follows from this and continuity of the functions.

Putting everything together, we get

$$\lim_{N\to\infty} S_N f(x) - f(x) = 0 \quad \square$$

4. Convolution

As mentioned above, $S_N(f)$ is the convolution of f and D_N. Let's study this notion of convolution in more detail:

Definition: If f and g are two integrable 2π periodic functions, then

$$(f \star g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy$$

Mnemonic: The sum is always x, namely $y + (x-y) = x - y + y = x$

Note: The two definitions are equivalent because if you let $u = x-y$ in the first integral, then $du = -dy$ and $y = x-u$ and therefore (ignoring the 2π for clarity)

$$\int_{-\pi}^{\pi} f(y)g(x-y)dy = \int_{x+\pi}^{x-\pi} f(x-u)g(u)(-du) = \int_{x-\pi}^{x+\pi} f(x-u)g(u)du$$
$$= \int_{-\pi}^{\pi} f(x-u)g(u)du$$

The last step follows from periodicity.

Immediate Properties:

(1) $f \star (g+h) = f \star g + f \star h$

(2) $(cf) \star g = c(f \star g) = f \star (cg)$

(3) $f \star g = g \star f$

(4) $(f \star g) \star h = f \star (g \star h)$

Fact: If f and g are continuous, then $f \star g$ is continuous

Proof:[3] Since g is continuous on $[-\pi, \pi]$, it is uniformly continuous on $[-\pi, \pi]$.

Let $M = \sup_x |f(x)|$

Let $\epsilon > 0$ be given. Then there is $\delta > 0$ such that if $|x - y| < \delta$ then $|g(x) - g(y)| < \epsilon$.

With the same δ, if $|x_1 - x_2| < \delta$, then (again ignoring the 2π)

$$\begin{aligned}
|(f \star g)(x_1) - (f \star g)(x_2)| =& \left| \int_{-\pi}^{\pi} f(y) g(x_1 - y) dy - \int_{-\pi}^{\pi} f(y) g(x_2 - y) dy \right| \\
\leq& \left| \int_{-\pi}^{\pi} f(y) \left(g(x_1 - y) - g(x_2 - y) \right) dy \right| \\
\leq& \int_{-\pi}^{\pi} \underbrace{|f(y)|}_{M} \underbrace{|g(x_1 - y) - g(x_2 - y)|}_{\epsilon} dy \\
\leq& 2\pi\epsilon M \checkmark
\end{aligned}$$

Notice: In the proof above, we never used continuity of f, just that f is bounded. In fact this is always true, $f \star g$ is always at least regular (continuour or smooth) as the more regular one of f and g. For

[3]The proof is taken from Prop 3.1 in Chapter 2 of Stein and Shakarchi

example:

Fact: If f is continuous and g is differentiable, then $f \star g$ is differentiable and

$$(f \star g)' = f \star (g')$$

Informally, this follows because

$$(f \star g)'(x) = \left(\int_{-\pi}^{\pi} f(y) g(x-y) dy \right)' = \int_{-\pi}^{\pi} f(y) g'(x-y) dy = f \star (g')$$

The differentiation is justified by taking difference quotients and using the Dominated Convergence Theorem (see Chapter 11)

How are convolutions related to Fourier series? Because of the following fact:

Fact: If f and g are continuous, then

$$\widehat{f \star g}(n) = \hat{f}(n) \hat{g}(n)$$

So the Fourier coefficient of $f \star g$ is the product of the Fourier coefficients of f and g, this is what makes convolution so nice!

Proof: Again, ignore the factor of 2π in the fourier coefficients and in the convolution

$$\begin{aligned}
\widehat{f \star g}(n) &= \int_{-\pi}^{\pi} (f \star g)(x) e^{-inx} dx \\
&= \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} f(y) g(x-y) dy \right) e^{-inx} dx \\
&\overset{\text{FUBINI}}{=} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(y) g(x-y) e^{-inx} dx dy \\
&= \int_{-\pi}^{\pi} f(y) \int_{-\pi}^{\pi} g(x-y) e^{-in(x-y)} e^{-iny} dx dy \\
&= \int_{-\pi}^{\pi} f(y) e^{-iny} \left(\int_{-\pi}^{\pi} g(x-y) e^{-in(x-y)} dx \right) dy \\
&= \int_{-\pi}^{\pi} f(y) e^{-iny} \left(\int_{-\pi}^{\pi} g(u) e^{-inu} du \right) dy \quad (\text{Use } u = x - y) \\
&= \left(\int_{-\pi}^{\pi} f(y) e^{-iny} dy \right) \left(\int_{-\pi}^{\pi} g(u) e^{-inu} du \right) \\
&= \hat{f}(n) \hat{g}(n)
\end{aligned}$$

(The same result is true if f and g are just integrable, but we would need an approximation theorem for that)

5. Optional: Convolution Intuition

Video: Convolution Intuition

Intuitively, $f \star g$ is the "multiplication" of f and g. This makes sense at least in terms of Fourier coefficients (see above).

Here is another way to think of it in terms of multiplication.

Question: What is the coefficient $h(2)$ of x^2 in

$$\left(a_0 + a_1 x + a_2 x^2\right)\left(b_0 + b_1 x + b_2 x^2\right)?$$

Multiplying out, the coefficient of x^2 becomes

$$h(2) = a_0 b_2 + a_1 b_1 + a_2 b_0 = \sum_{k=0}^{2} a_k b_{2-k}$$

Compare this with

$$(f \star g)(x) = \int_{-\pi}^{\pi} f(y) g(x-y) dy \text{ “ = ” } \sum f(y) g(x-y)$$

Which is also a sum of terms of the form $a_k \ b_{x-k}$

So in some sense $(f \star g)(x)$ is the $x-$th coefficient of f times g, if you think of f and g as polynomials.

In this sense, convolution becomes sort of like multiplication of f and g.