LECTURE 9: CAUCHY SEQUENCES; SUBSEQUENCES

1. Cauchy Sequences

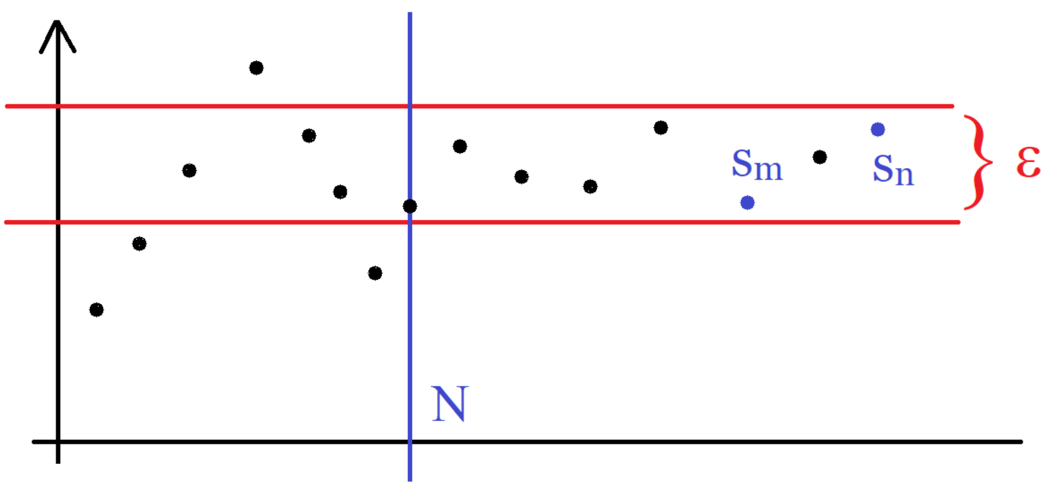
Video: Cauchy Sequences

Last time: Discussed the notion of a Cauchy sequence, which are just sequences that are getting close to *each other* (instead of closer to s).

Definition:

(s_n) is a **Cauchy sequence** if for all $\epsilon > 0$ there is N such that if $m, n > N$, then

$$|s_n - s_m| < \epsilon$$



Date: Tuesday, September 28, 2021.

Just like for convergent sequences, Cauchy sequences are bounded.

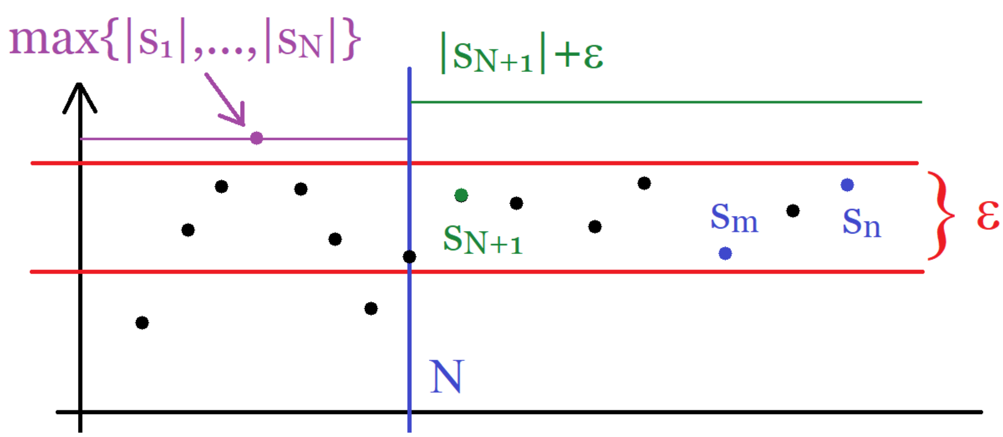
Cauchy sequences are bounded

If (s_n) is Cauchy, then (s_n) is bounded

Proof: Let $\epsilon > 0$ be arbitrary, then there is (an integer) N such that if $m, n > N$, then $|s_m - s_n| < \epsilon$

But *in particular* with $m = N + 1 > N$, we get that if $n > N$, then $|s_{N+1} - s_n| < \epsilon$. So if $n > N$, we have:

$$|s_n| = |s_n - s_{N+1} + s_{N+1}| \leq |s_n - s_{N+1}| + |s_{N+1}| < \underbrace{\epsilon + |s_{N+1}|}_{\text{FIXED}}$$



$$\text{Now let} \quad M = \max\{|s_1|, |s_2|, \ldots, |s_N|, \epsilon + |s_{N+1}|\} > 0$$

Case 1: If $n \leq N$, then:

$$|s_n| \leq \max\{|s_1|, |s_2|, \ldots, |s_N|\} \leq M \checkmark$$

Case 2: If $n > N$, then

$$|s_n| \leq \epsilon + |s_{N+1}| \leq M \checkmark$$

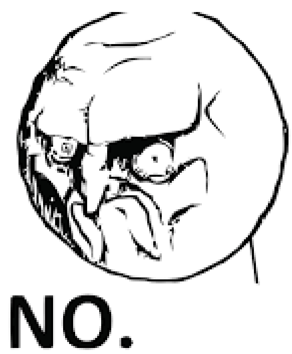
So in either case, for all n, $|s_n| \leq M$, so (s_n) is bounded $\square$

2. Completeness

Video: Completeness

Last time: We've seen that *if* (s_n) is convergent, then it is Cauchy.

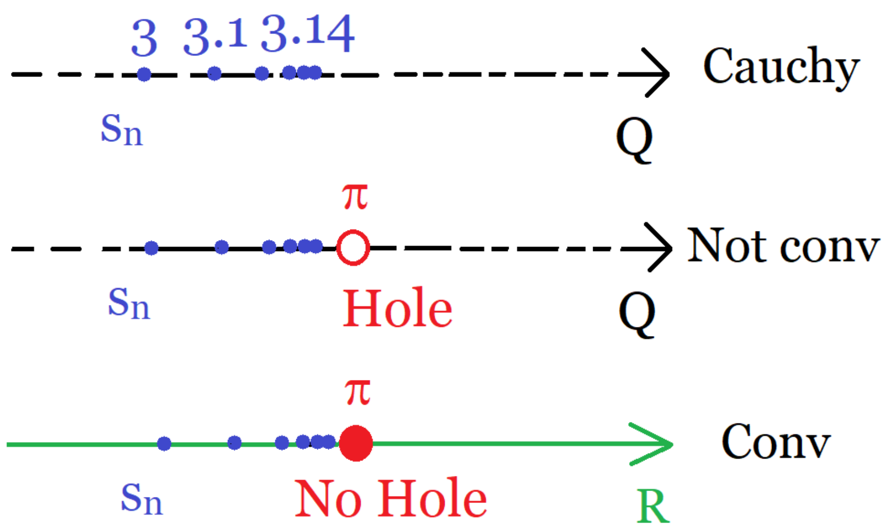
But what about the converse: If (s_n) is Cauchy, then is it convergent?



Analogy: Just because you see a large crowd (Cauchy), it doesn't mean that the crowd is going somewhere

Example: Pretend our universe is $\mathbb{Q}$ and you don't know that real numbers exist. Now look at the following sequence:

$$(s_n) = (3, 3.1, 3.14, 3.141, 3.1415, \dots)$$



This sequence is Cauchy, since all the terms are getting closer to each other. But does (s_n) converge (in $\mathbb{Q}$)? **NO!** Because if (s_n) converges, then it must converge to π, which is not in $\mathbb{Q}$. So in our universe, the limit does not exist.

But in $\mathbb{R}$, the answer is **YES**: Cauchy sequences in $\mathbb{R}$ are convergent, which explains *yet again* why real numbers are so great.

$\mathbb{R}$ is complete

If (s_n) is a Cauchy sequence in $\mathbb{R}$, then (s_n) converges

Proof: Since we don't know what the limit is, let's use the lim sup squeeze theorem from last time. Namely, let's show that:

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n$$

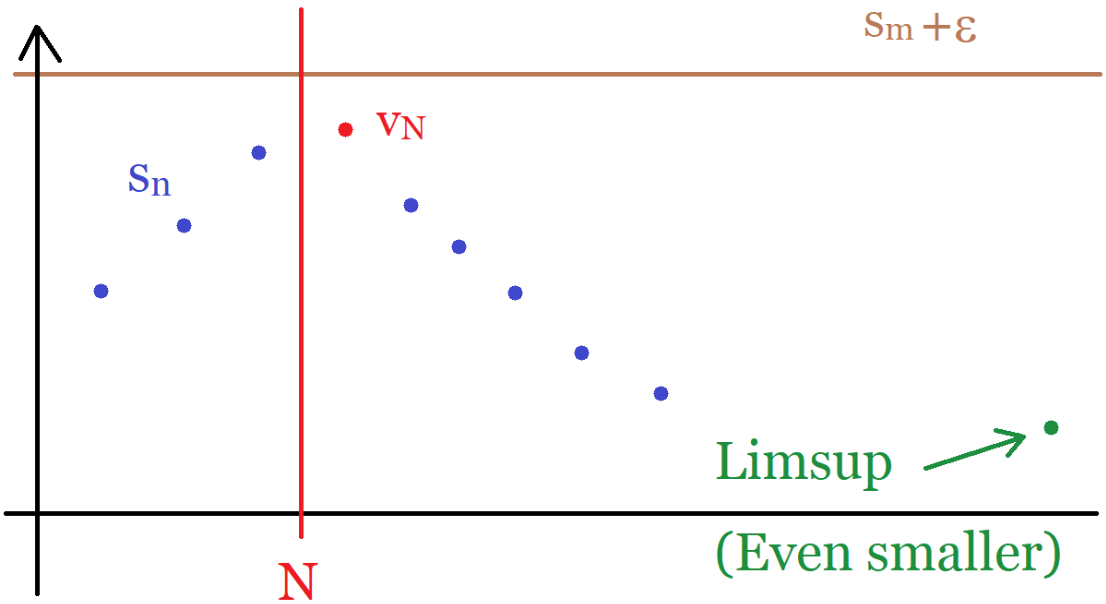
STEP 1: Let $\epsilon > 0$ be arbitrary

Since (s_n) is Cauchy, there is N such that if $m, n > N$, then:

$$|s_n - s_m| < \epsilon \Rightarrow s_n - s_m < \epsilon \Rightarrow s_n < s_m + \epsilon$$

Since this is true for *all* $n > N$, taking the sup over n, we get:

$$v_N =: \sup\{s_n \mid n > N\} \leq s_m + \epsilon$$



But from last time, since v_N is decreasing with limit $\limsup_{n\to\infty} s_n$, so for all $n > N$,

$$\limsup_{n\to\infty} s_n \leq v_N \leq s_m + \epsilon \Rightarrow \limsup_{n\to\infty} s_n \leq s_m + \epsilon$$

$$\text{Therefore} \quad s_m \geq \left(\limsup_{n\to\infty} s_n\right) - \epsilon$$

STEP 2: Since this is true for all $m > N$, taking the inf over m, we get:

$$u_N =: \inf\{s_m \mid m > N\} \geq \left(\limsup_{n\to\infty} s_n\right) - \epsilon$$

Since u_N is increasing in N with limit $\liminf_{n\to\infty} s_n$, we get for all $m > N$:

$$\liminf_{n\to\infty} s_n \geq u_N \geq \left(\limsup_{n\to\infty} s_n\right) - \epsilon \Rightarrow \liminf_{n\to\infty} s_n \geq \left(\limsup_{n\to\infty} s_n\right) - \epsilon$$

And since ϵ was arbitrary, we obtain

$$\liminf_{n\to\infty} s_n \geq \limsup_{n\to\infty} s_n$$

But since also $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n$, we get $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$ Therefore, by the limsup squeeze theorem, (s_n) must converge to some s. □

Definition:

A space is **complete** if every Cauchy sequence in it converges

Examples: $\mathbb{R}$ (just shown), but also $\mathbb{R}^n$ and $\mathbb{Z}$ (see HW)

Non-Example: $\mathbb{Q}$

Intuitively: A complete space doesn't have holes, just like $\mathbb{R}$ doesn't have holes, but $\mathbb{R}$ does.

Fun Fact 1:

Can always complete an incomplete space, like completing $\mathbb{Q}$ to get $\mathbb{R}$

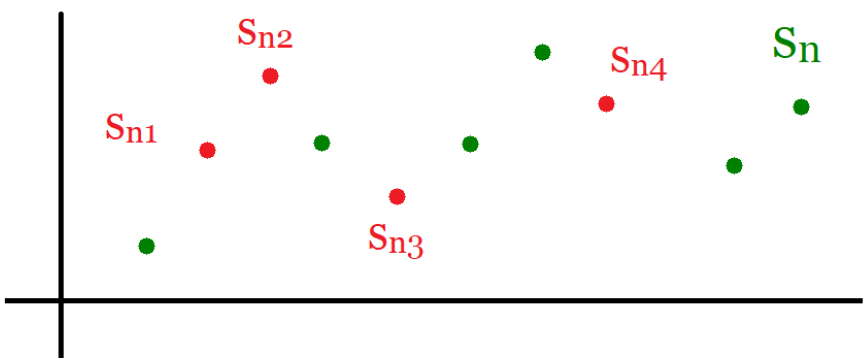
Fun Fact 2:

It is possible to construct $\mathbb{R}$ just by using Cauchy sequences of rational numbers! So a real number is just a (class of) sequences of rational numbers, just like a rational number is just a (class of) pairs of integers

3. Subsequences

Video: What is a Subsequence?

Suppose we have a sequence (s_n). Think of (s_n) as a train that goes through cities (such as Peyamgeles, Liouville, Sup Francisco, or Infianapolis).



Then a subsequence (s_{n_k}) is an express train, that which goes through the same cities as (s_n), but skips some cities.

In the picture above, s_{n_1} (the first express stop) is the second city, s_{n_2} is the 3rd city, s_{n_3} is the 5th city, and s_{n_4} is the 8th city.

Definition:

A **subsequence** of (s_n) is a sequence of the form (s_{n_k}) with $n_1 < n_2 < \dots$, where for every k, you associate a value s_{n_k} of (s_n)

So every express stop s_{n_k} has to be one of the original stops of (s_n), but s_{n_k} can skip some of the cities; And the condition $n_1 < n_2 < \dots$ says that the second stop s_{n_2} comes *after* the first stop s_{n_1}, and so on. In other words, the express train is going forwards, not backwards

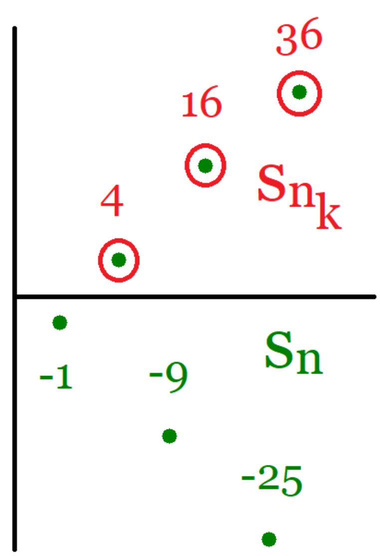
Example 1:

Let (s_n) be the sequence:

$$s_n = (-1)^n n^2 = (-1, 4, -9, 16, -25, \dots)$$

Define the subsequence (s_{n_k}) by $s_{n_k} = k^{th}$ positive term of (s_n)

That is, just look at the positive terms of (s_n) and skip the negative ones. Or, in other words, just look at every second term of s_n



$$
\begin{aligned}
s_{n_1} &= s_2 = 4 \\
s_{n_2} &= s_4 = 16 \\
s_{n_3} &= s_6 = 36
\end{aligned}
$$

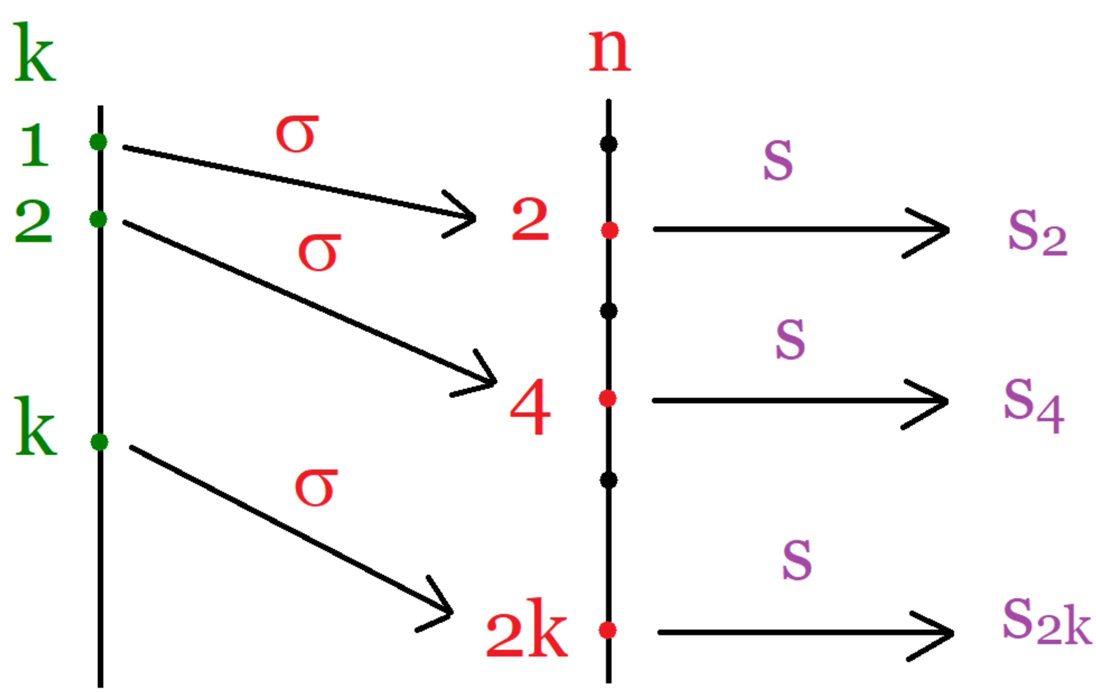
And, following that pattern, you might guess that:

$$s_{n_k} = s_{2k} = (2k)^2 = 4k^2$$

Remark: We can express this as a composition of two functions: if you define $\sigma(k) = 2k$, then we have:

$$s_{n_k} = s_{2k} = s_{\sigma(k)}$$

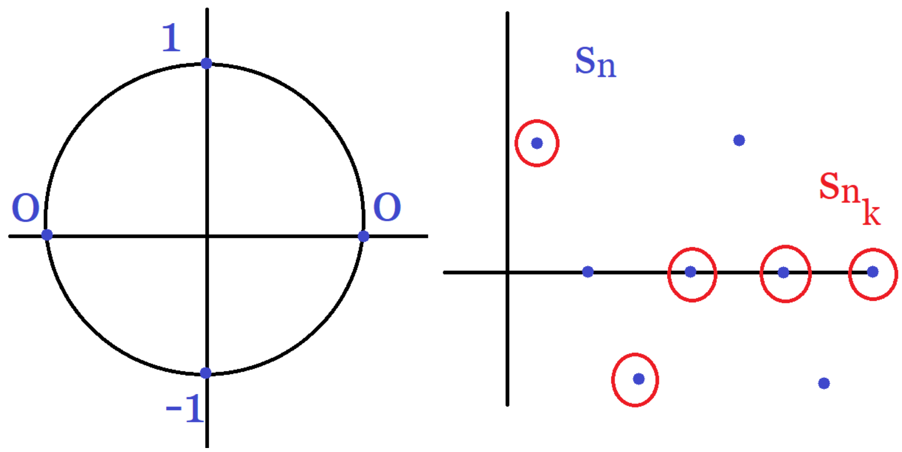
So $\sigma(k)$ takes k as an input and gives us $n = 2k$, and our sequence s_n takes $n = 2k$ as an input and spits out s_{2k}.



Example 2:

Let (s_n) be the sequence:

$$s_n = \sin\left(\frac{\pi n}{2}\right) = (1, 0, -1, 0, 1, 0, -1, 0, \cdots)$$



One example of a subsequence is

$$(s_{n_k}) = (1, -1, 0, 0, 0, \dots)$$

Notice that, even though the original sequence (s_n) doesn't converge, the subsequence (s_{n_k}) converges! And in fact we'll see next time that any (bounded) sequence will always have a convergent subsequence!

But, if (s_n) converges to s, then any subsequence must also converge to s. This makes sense: If a train leads you to a final destination, then any express train must also go to that final destination.

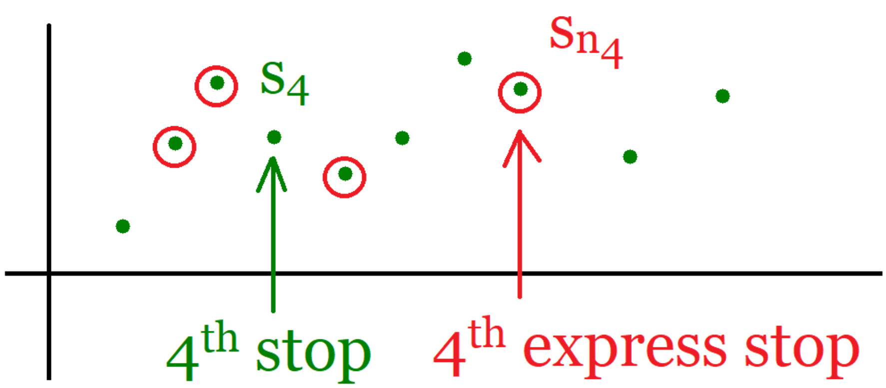
Convergence Fact:

If $\lim_{n\to\infty} s_n = s$, then $\lim_{k\to\infty} s_{n_k} = s$ as well.

Note: The proof of this relies on the following little fact:[1]

Little Fact:

For all k, $n_k \geq k$



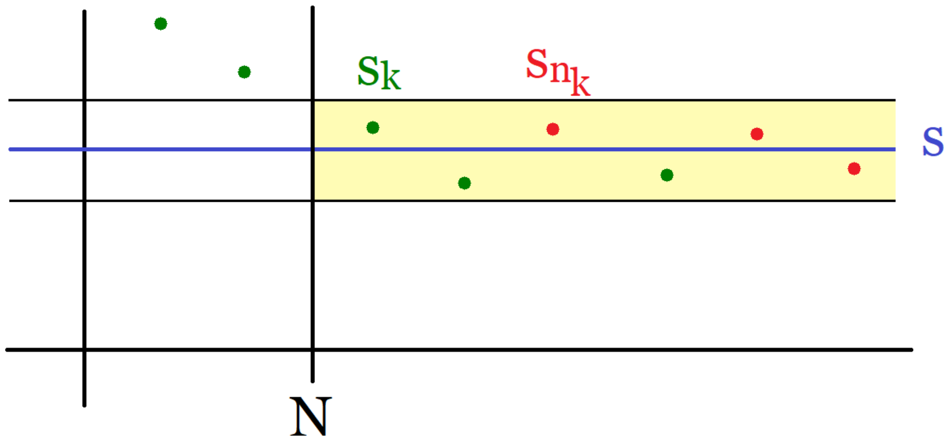
In other words, the k-th express stop must be after the k-th non-express stop. Otherwise, why would it be called an *express* train?

Proof of Convergence Fact:

Let $\epsilon > 0$ be given, since (s_n) converges to s, there is N such that for all $n > N$, then $|s_n - s| < \epsilon$

Now for the same N, if $k > N$, then $n_k \geq k > N$, so $k > N$ and therefore $|s_{n_k} - s| < \epsilon$, so s_{n_k} converges to s $\square$

[1]The proof is by induction: The inductive step is $n_{k+1} > n_k \geq k$, so $n_{k+1} > k$ and therefore $n_{k+1} \geq k + 1$ since k is an integer. For example, if $n_{k+1} > 6$ then $n_{k+1} \geq 7$



4. Inductive Construction 1

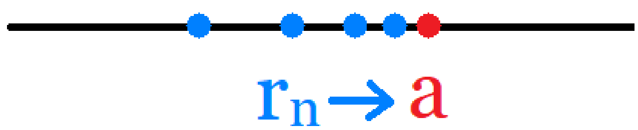
Video: Inductive Construction 1

Now that we got the basics down, let's cover an important technique called an inductive construction (of subsequence). This technique will be used over and over again in this course.

Note: Here I give a somewhat simplified presentation of the book and the video.

Example 3:

If $a \in \mathbb{R}$, show that there is an increasing sequence (r_n) of rational numbers that converges to a



Note: Although the fact itself is very neat, it's really the technique of the proof that's important here!

Proof: Our goal is to construct an increasing sequence (r_n) with the property that for every $n \in \mathbb{N}$

$$a - \frac{1}{n} < r_n < a$$

Idea: First construct r_1 and, given r_n, construct r_{n+1}

STEP 1: Base Case:

Construct r_1: Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a rational number r_1 with $a - 1 < r_1 < a$ ✓

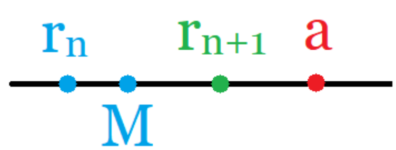
STEP 2: Inductive step:

Suppose you found $r_1 < r_2 < \cdots < r_n$ such that $a - \frac{1}{k} < r_k < a$ for all $k = 1, \ldots, n$

Goal: Find r_{n+1} with $r_{n+1} > r_n$ and $a - \left(\frac{1}{n+1}\right) < r_{n+1} < a$

Note: It is not *quite* enough to repeat the density argument above, since this doesn't guarantee that $r_{n+1} > r_n$

To get around this, let $M = \max\left\{r_n, a - \frac{1}{n+1}\right\}$



Then since $\mathbb{Q}$ is dense in $\mathbb{R}$ there is r_{n+1} such that $M < r_{n+1} < a$, so in particular, since M is the max, $r_{n+1} > r_n$ ✓and $a - \frac{1}{n+1} < r_{n+1} < a$.

STEP 3: Therefore, by this inductive construction, we have found an increasing sequence (r_n) such that $a - \frac{1}{n} < r_n < a$, and lastly, by the squeeze theorem, we get $\lim_{n\to\infty} r_n = a$ $\square$