LECTURE 9: FUNCTIONS OF SEVERAL VARIABLES

Welcome to the heart of *multi*-variable calculus, because today we'll cover functions of *several* variables.

1. The basics

So far our functions depended only on *one* variable, like $f(x) = x^2$, but from now on our functions will have two or more variables:

1.1. **Examples:**

Example 1:

Let $f(x, y) = 2x^2 + 3xy + y^2$

Find $f(1, 2)$ and $f(-1, 0)$

For $f(1, 2)$, we have $x = 1$ and $y = 2$ and so

$$f(1, 2) = 2(1)^2 + 3(1)(2) + (2)^2 = 2 + 6 + 4 = 12$$

$$f(-1, 0) = 2(-1)^2 + 3(-1)(0) + (0)^2 = 2$$

This is very useful in applications, because real life phenomena depend on many variables; think functions of Position and Time, or Temperature and Pressure.

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1.2. **Domains:** The nice thing is that a lot of concepts from calculus, such as domain, range, and graphs, generalize easily to higher dimensions:

Example 2:

Sketch the domains of the following functions:

(a) $f(x, y) = \ln(4 - x^2 - y^2)$

(b) $f(x, y) = \sqrt{1 - |x| - |y|}$

Definition:

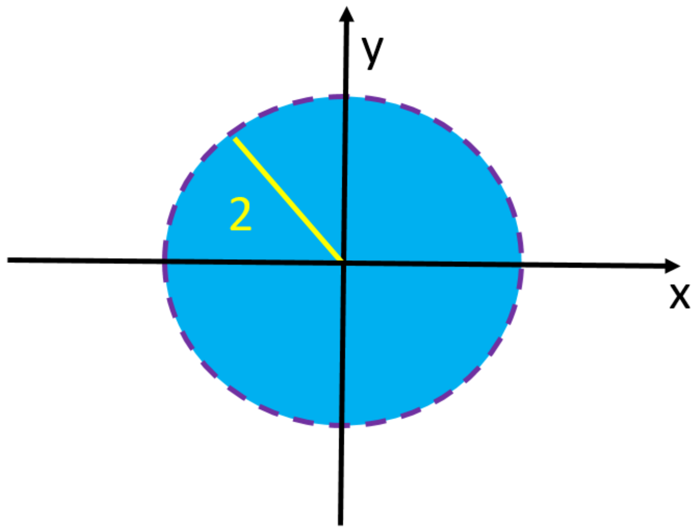
The **domain** of a function f is the set of numbers (x, y) for which $f(x, y)$ is defined

(Just like in 1 dimension where the domain of $f(x) = \sqrt{x}$ is $[0, \infty)$)

(a) Since $\ln(x)$ is only defined for **positive** x, the the domain of $\ln(4 - x^2 - y^2)$ is

$$4 - x^2 - y^2 > 0 \Rightarrow x^2 + y^2 < 4$$

Which is the inside of the disk centered at $(0, 0)$ and radius 2 (without the boundary)



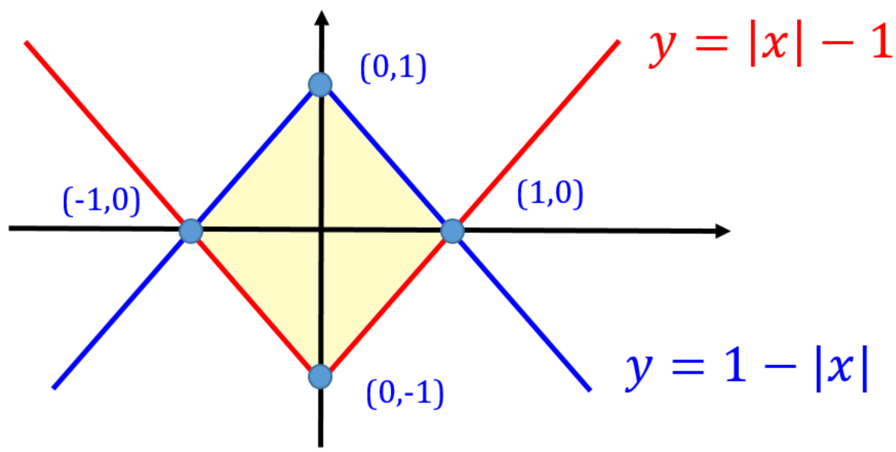
Careful Note: In 1 dimensions, the domain is an interval like $[0, \infty)$, but in 2 dimensions, the domain is a 2D region.

(b) $f(x, y) = \sqrt{1 - |x| - |y|}$

Since you cannot take square roots of negative numbers, here we need:

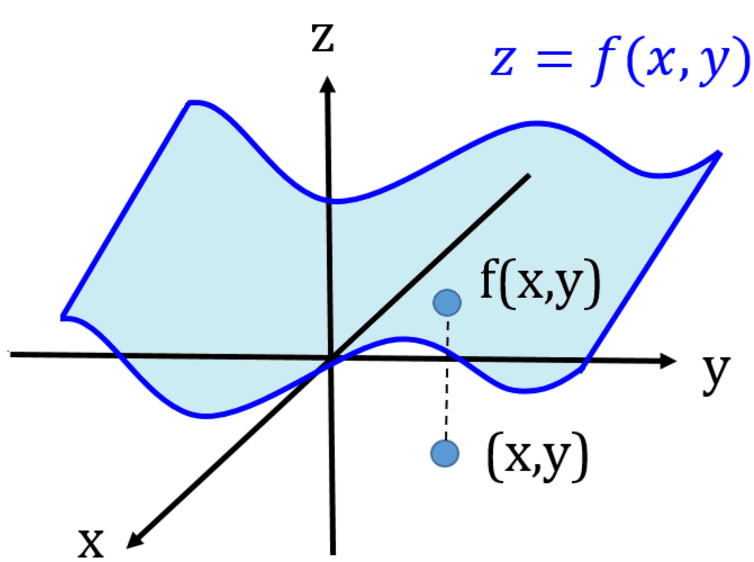
$$\begin{aligned} & 1 - |x| - |y| \geq 0 \\ \Rightarrow & |x| + |y| \leq 1 \\ \Rightarrow & |y| \leq 1 - |x| \\ \Rightarrow & -(1 - |x|) \leq y \leq 1 - |x| \\ \Rightarrow & |x| - 1 \leq y \leq 1 - |x| \end{aligned}$$

(Here we used that $|A| \leq B \Leftrightarrow -B \leq A \leq B$)



So here the domain is a **square** with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$

1.3. **Graphs:** Similarly, functions in several variables have graphs. The only difference is that in one dimension, graphs are curves, but in 2 dimensions, graphs are surfaces:



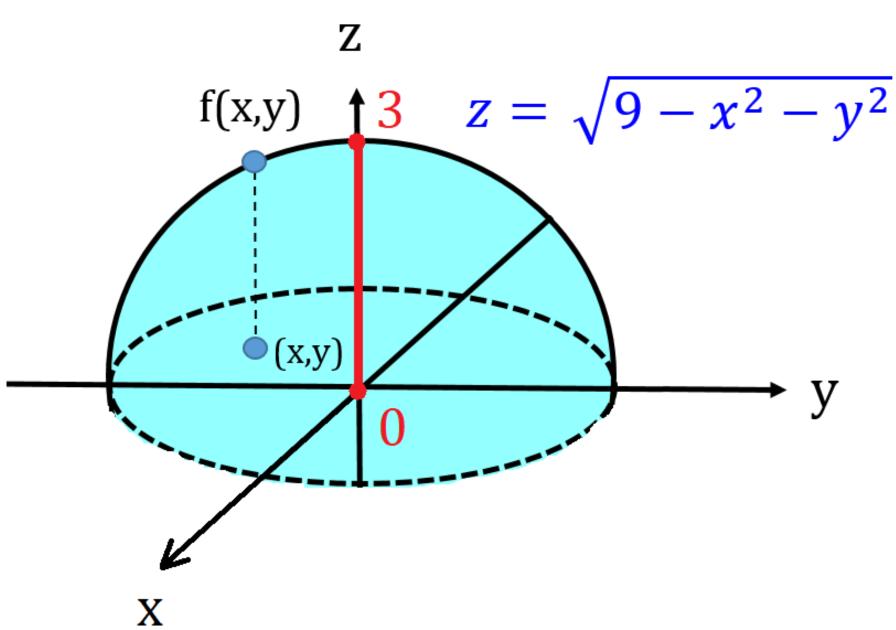
Example 3:

Sketch the graph of $f(x, y) = \sqrt{9 - x^2 - y^2}$

This is actually an object that we've seen before! Notice that

$$z = \sqrt{9 - x^2 - y^2} \Rightarrow z^2 = 9 - x^2 - y^2 \Rightarrow x^2 + y^2 + z^2 = 9$$

The graph is *not quite* a sphere; because $z = \sqrt{9 - x^2 - y^2} \geq 0$, the graph is actually an upper hemisphere centered at $(0, 0, 0)$ and radius 3.



1.4. **Range:** Related to domains is the concept of range. which is the set of all possible values of the function.

Example 4:

Find the range of $f(x, y) = \sqrt{9 - x^2 - y^2}$

Definition:

The **range** of a function f is the set of all possible values of $z = f(x, y)$

Based on the graph above, notice that $z = \sqrt{9 - x^2 - y^2}$ goes from 0 to 3, so the range is $[0, 3]$

Note: Here the range is in fact an interval, just like in 1 dimensions.

1.5. **Limits: (optional).** Finally, we can even take limits of functions, just like in 1 dimensions.

Example 5:

Find the following limit:

$$\lim_{(x,y)\to(1,2)} \frac{2x + 4y}{x^2 + y^2}$$

$$\lim_{(x,y)\to(1,2)} \frac{2x + 4y}{x^2 + y^2} = \frac{2(1) + 4(2)}{1^2 + 2^2} = \frac{10}{5} = 2$$

Interpretation:

As (x, y) goes to $(1, 2)$, $f(x, y) = \frac{2x+4y}{x^2+y^2}$ gets closer and closer to 2

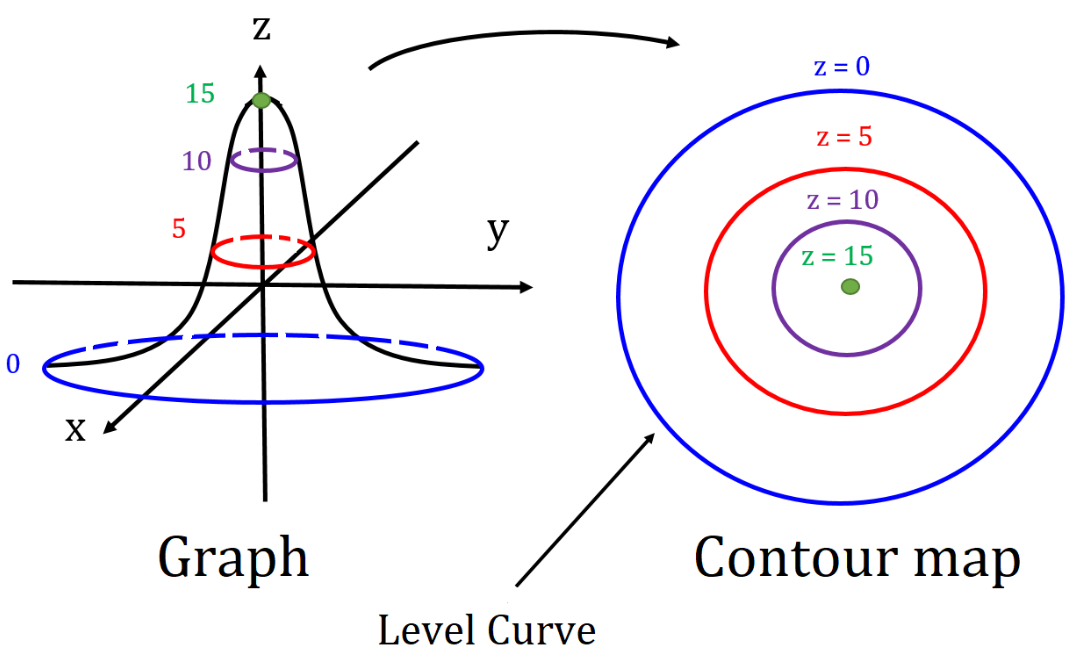
Note: Using this, we can define f is **continuous**, which intuitively means that the graph of f has no jumps (see section 14.2 if interested)

2. Level Curves

More importantly, we can talk about level curves, which is a useful tool for graphing functions in 3D and even 4D.

Example 6: (Motivation)

Consider the function $f(x, y)$ with the following graph:



Then we can look at all the points that are at a given height, for example all the points that are at height 10 or 15.

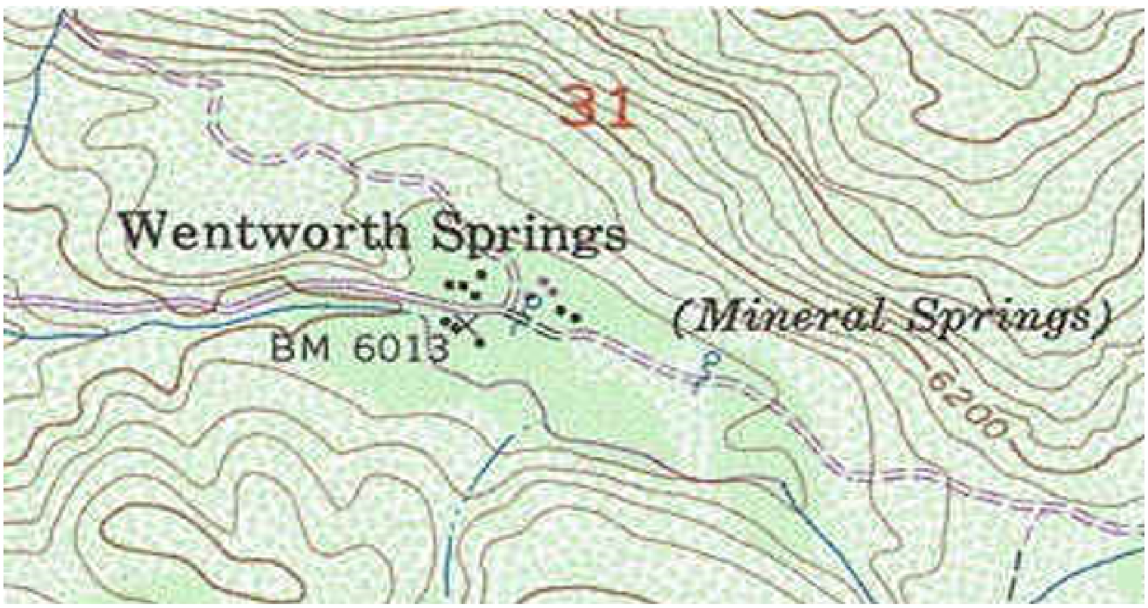
Then the curves that we get are called **level curves** and the picture that we get is called a **contour map**.

Definition:

Level Curves are curves of the form $z =$ Some Number

Note: Think of the level curves like slicing the graph horizontally, and the contour map like a birds-eye view of the graph.

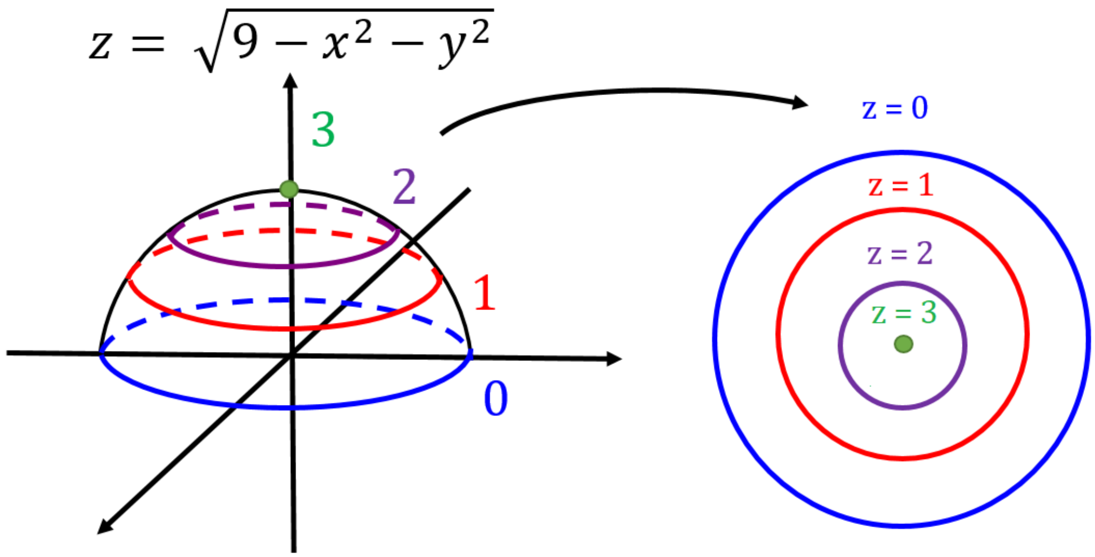
Application: Level curves such as the one below are used in maps to draw a 2D representation of a mountain.



Example 7:

Let $f(x, y) = \sqrt{9 - x^2 - y^2}$

Sketch the level curves $z = 0, z = 1, z = 2, z = 3$



$\mathbf{z = 0} \Rightarrow \sqrt{9 - x^2 - y^2} = 0 \Rightarrow 9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9$

$\mathbf{z = 1} \Rightarrow \sqrt{9 - x^2 - y^2} = 1 \Rightarrow 9 - x^2 - y^2 = 1 \Rightarrow x^2 + y^2 = 8$

$\mathbf{z = 2} \Rightarrow \sqrt{9 - x^2 - y^2} = 2 \Rightarrow 9 - x^2 - y^2 = 4 \Rightarrow x^2 + y^2 = 5$

$\mathbf{z = 3} \Rightarrow \sqrt{9 - x^2 - y^2} = 3 \Rightarrow 9 - x^2 - y^2 = 9 \Rightarrow x^2 + y^2 = 0 \Rightarrow (0, 0)$

So the level curves $z = 0, 1, 2, 3$ are circles centered at $(0, 0)$ and radii $3, \sqrt{8}, \sqrt{5}$, and 0 (point) respectively.

What's great about level curves is that they give us a 2D representation of a 3D figure, which is especially important in higher dimensions (see below).

Not only that, we can use level curves to sketch surfaces.

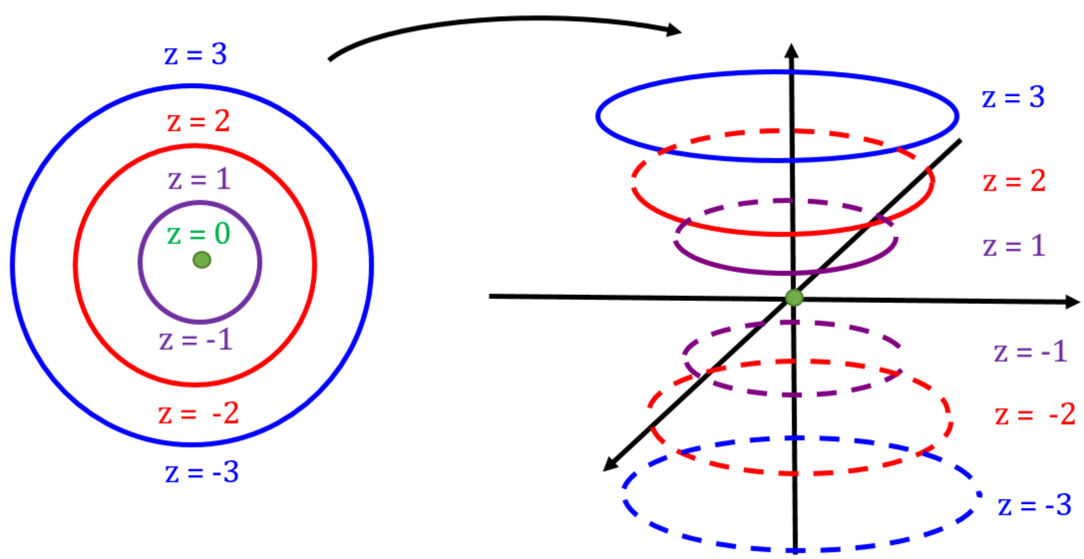
Example 8: (extra practice)

Use level curves to sketch the surface $x^2 + y^2 = z^2$

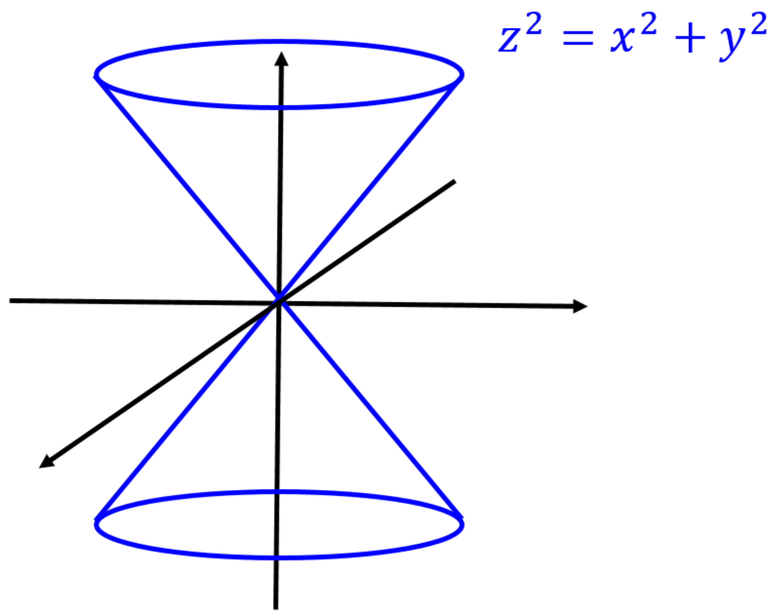
Unlike the previous example, z can also be negative, so let's also sketch some negative level curves (think oceans)

$$\begin{aligned}
\mathbf{z} = \mathbf{3} &\Rightarrow x^2 + y^2 = 3^2 \\
\mathbf{z} = \mathbf{2} &\Rightarrow x^2 + y^2 = 2^2 \\
\mathbf{z} = \mathbf{1} &\Rightarrow x^2 + y^2 = 1^2 \\
\mathbf{z} = \mathbf{0} &\Rightarrow x^2 + y^2 = 0^2 \Rightarrow (0,0) \\
\mathbf{z} = -\mathbf{1} &\Rightarrow x^2 + y^2 = (-1)^2 \Rightarrow x^2 + y^2 = 1 \\
\mathbf{z} = -\mathbf{2} &\Rightarrow x^2 + y^2 = (-2)^2 \Rightarrow x^2 + y^2 = 4 \\
\mathbf{z} = -\mathbf{3} &\Rightarrow x^2 + y^2 = (-3)^2 \Rightarrow x^2 + y^2 = 9
\end{aligned}$$

So here the level curves $z = 3, 2, 1, 0, -1, -2, -3$ are circles centered at $(0, 0)$ and radius $3, 2, 1, 0, 1, 2, 3$ respectively.



Therefore, connecting the level curves, we see that the surface $z^2 = x^2 + y^2$ is a **cone**:



3. Level Surfaces

What about functions of 3 variables, of the form $f(x, y, z)$? The beautiful thing is that everything that we learned about 2 variables generalizes to 3 variables as well, except that now level curves become level *surfaces*.

Example 9:

Let $w = f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ (hypercone)

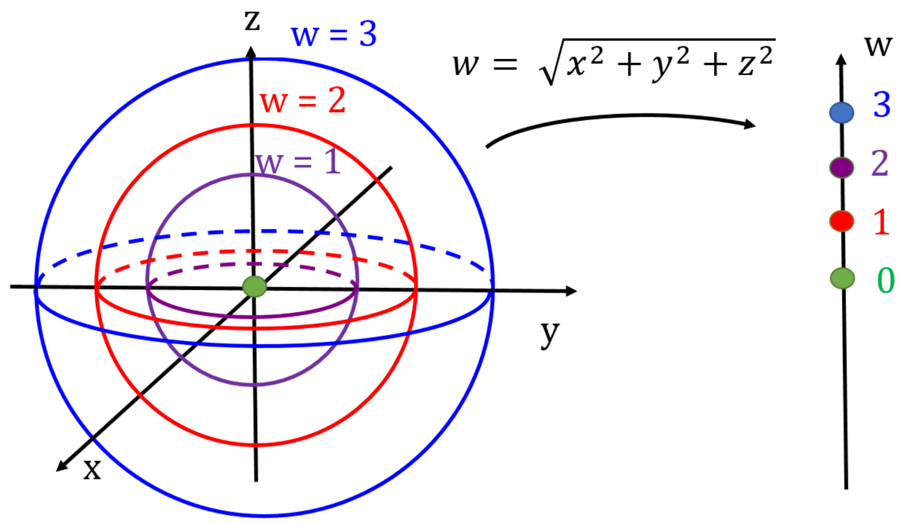
Sketch the level surfaces $w = 0, w = 1, w = 2, w = 3$

Definition:

A **level surface** is a surface of the form $w =$ Some number

$$\mathbf{w} = \mathbf{0} \Rightarrow \sqrt{x^2 + y^2 + z^2} = 0 \Rightarrow x^2 + y^2 + z^2 = 0 \Rightarrow (0,0,0)$$
$$\mathbf{w} = \mathbf{1} \Rightarrow \sqrt{x^2 + y^2 + z^2} = 1 \Rightarrow x^2 + y^2 + z^2 = 1$$
$$\mathbf{w} = \mathbf{2} \Rightarrow \sqrt{x^2 + y^2 + z^2} = 2 \Rightarrow x^2 + y^2 + z^2 = 4$$
$$\mathbf{z} = \mathbf{3} \Rightarrow \sqrt{x^2 + y^2 + z^2} = 3 \Rightarrow x^2 + y^2 + z^2 = 9$$

So the level surfaces $w = 0, 1, 2, 3$ are spheres centered at $(0, 0, 0)$ and radii $0, 1, 2, 3$ respectively.



Why important: We cannot draw the surface $w = \sqrt{x^2 + y^2 + z^2}$ because it's in 4 dimensions, but this process allows us to visualize it. The level surfaces tell us that this is a (hyper)cone whose slices are spheres, just like for the regular cone where the slices are circles. In some sense, this gives us a glimpse in the 4th dimension!

Note: For the hypersphere $x^2 + y^2 + z^2 + w^2 = r^2$, it's even freakier! It's a sphere (in 4 dimensions) whose level surfaces are spheres (in 3 dimensions). So it's like cutting an orange whose slices are oranges!

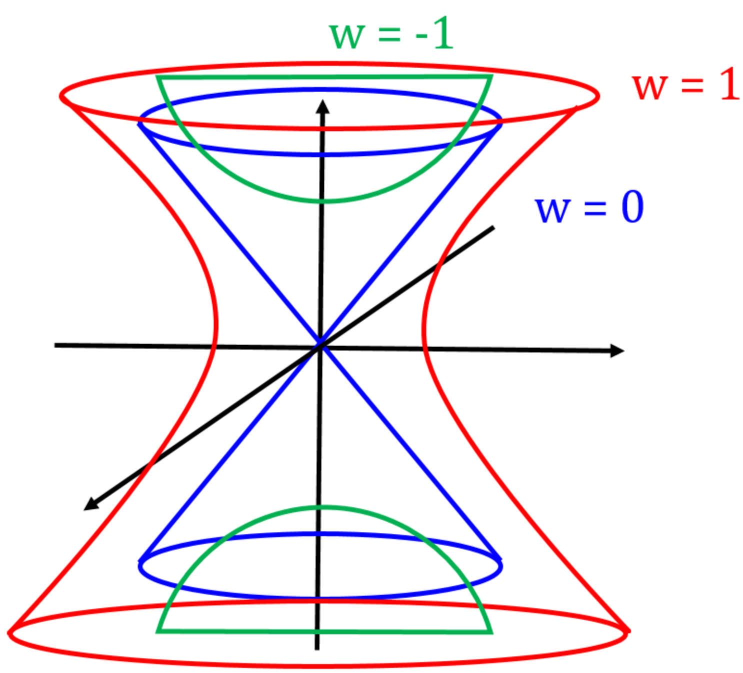
Similarly a 5−dimensional sphere is an orange whose slices are (4-dimensional) oranges, wow!

Example 10 : (extra practice)

Let $w = f(x, y, z) = x^2 + y^2 - z^2$

Sketch the level surfaces $w = -1, w = 0, w = 1$

$$\mathbf{w} = -\mathbf{1} \Rightarrow x^2 + y^2 - z^2 = -1 \Rightarrow -x^2 - y^2 + z^2 = 1 \text{ (Two cups)}$$
$$\mathbf{w} = \mathbf{0} \Rightarrow x^2 + y^2 - z^2 = 0 \Rightarrow z^2 = x^2 + y^2 \text{ (Cone)}$$
$$\mathbf{w} = \mathbf{1} \Rightarrow x^2 + y^2 - z^2 = 1 \text{ (Dress)}$$



So it's a weird surface in 4 dimensions whose positive slices are two cups, the zero-slice is a cone, and negative slices are dresses. What in the world is this???