LECTURE 9: ORTHOGONALITY

We can get very powerful properties of Fourier series if we examine them under a more abstract viewpoint.

1. Orthogonality

Consider the space of 2π periodic integrable functions, which we can now equip with an inner product:

Definition:

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$$

It's a function analog of the dot product for complex numbers

$$(x_1, x_2, \cdots, x_n) \cdot (y_1, y_2, \cdots, y_n) = \sum_{k=1}^{n} x_k \overline{y_k}$$

(since an integral is just a big sum)

We can then define the length of a function as

$$\|f\|^2 = (f,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 \, dx$$

Notation: $e_n(x) = e^{inx}$

What makes the e_n so special is that

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Fact: The family $\{e_n\}_{n=-\infty}^{\infty}$ is **orthonormal**, that is

$$(e_n, e_m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Why?

$$(e_n, e_m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\overline{e^{imx}}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}dx$$

If $n \neq m$, this integral is 0 (see the first lecture on Fourier series), and if $n = m$ this integral is 1 □

We will see today that orthogonality lies in the heart of a lot of powerful results about Fourier series.

Notation:

$$a_n =: \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = (f, e_n)$$

And in particular, we get

$$S_N(f) = \sum_{n=-N}^{N} a_n e_n \qquad \text{(linear combo of } e_n\text{)}$$

And Fourier analysis is the study of

$$f - S_N(f) = f - \sum_{n=-N}^{N} a_n e_n$$

2. Orthogonal Decomposition

Geometrically, $S_N(f)$ is just the orthogonal projection of f on the span of $\{e_n\}_{n=-N}^{N}$ (see picture in lecture), and in fact:

Fact: For any complex numbers b_n,

$$(f - S_N(f)) \perp \sum_{n=-N}^{N} b_n e_n$$

Fix m and calculate

$$(f - S_N(f), e_m) = \left(\left(f - \sum_{n=-N}^{N} a_n e_n\right), e_m\right) = \underbrace{(f, e_m)}_{a_m} - \sum_{n=-N}^{N} a_n \underbrace{(e_n, e_m)}_{0 \text{ or } 1}$$
$$= a_m - a_m = 0$$

And the result follows by taking linear combinations of the e_m □

Recall: [Pythagorean Theorem]

$$\text{If } u \perp v \text{ then } \|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Theorem: [Orthogonal Decomposition]

$$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{n=-N}^{N} |a_n|^2$$

Might not look like much, but we'll see soon why this is so useful.

Proof:

$$f = (f - S_N(f)) + S_N(f) = (f - S_N(f)) + \sum_{n=-N}^{N} a_n e_n$$

And using the fact above and the Pythagorean Theorem, we get

$$\|f\|^2 = \|f - S_N(f)\|^2 + \left\| \sum_{n=-N}^{N} a_n e_n \right\|^2$$

And since the e_n are orthonormal, applying the Pythagorean theorem many times to the second term, we get

$$\left\| \sum_{n=-N}^{N} a_n e_n \right\|^2 = \sum_{n=-N}^{N} \|a_n e_n\|^2 = \sum_{n=-N}^{N} |a_n|^2 \underbrace{\|e_n\|^2}_{1} = \sum_{n=-N}^{N} |a_n|^2$$

Which gives the desired result above □

Corollary: [Bessel's Inequality]

$$\sum_{n=-\infty}^{\infty} |a_n|^2 \leq \|f\|^2$$

Proof:

$$\|f\|^2 = \underbrace{\|f - S_N(f)\|^2}_{\geq 0} + \sum_{n=-N}^{N} |a_n|^2 \geq \sum_{n=-N}^{N} |a_n|^2$$

Hence for all N, we have

$$\sum_{n=-N}^{N} |a_n|^2 \leq \|f\|^2$$

And taking the limit as N goes to ∞ gives the result □

Corollary: [Riemann Lebesgue Lemma]
If f is 2π periodic and integrable, then $\lim_{n \to \pm\infty} \hat{f}(n) = 0$

That is, the Fourier coefficients of an integrable function decay to 0.

Note: Taking real and imaginary parts, we get

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0 \text{ and } \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin(nx) = 0$$

This is the result we needed last time

Why? From Bessel's Inequality we get

$$\sum_{n=-\infty}^{\infty} |a_n|^2 \leq \|f\|^2 < \infty$$

So since $\sum_{n=-\infty}^{\infty} |a_n|^2$ converges this implies $\lim_{n\to\pm\infty} a_n = 0$ □

3. Best Approximation Lemma

Corollary: [Best Approximation Lemma]

$$\text{For any } c_n \text{ we have } \|f - S_N(f)\| \leq \left\| f - \sum_{n=-N}^{N} c_n e_n \right\|$$

Equality holds precisely when $c_n = a_n$ for all n in $[-N, N]$

This is really interesting! It says that the trigonometric polynomial that is closest to f is precisely the Fourier series!

Proof:

$$\begin{aligned} f - \sum_{n=-N}^{N} c_n e_n &= f - S_N(f) + S_N(f) - \sum_{n=-N}^{N} c_n e_n \\ &= f - S_N(f) + \sum_{n=-N}^{N} a_n e_n - \sum_{n=-N}^{N} c_n e_n \\ &= f - S_N(f) + \sum_{n=-N}^{N} (a_n - c_n) e_n \end{aligned}$$

Therefore from the Pythagorean Theorem and orthogonality, we get

$$\left\| f - \sum_{n=-N}^{N} c_n e_n \right\|^2 = \| f - S_N(f) \|^2 + \underbrace{\left\| \sum_{n=-N}^{N} (a_n - c_n) e_n \right\|^2}_{\geq 0} \geq \| f - S_N(f) \|^2 \checkmark$$

And equality holds when

$$\left\| \sum_{n=-N}^{N} (a_n - c_n) e_n \right\|^2 = \sum_{n=-N}^{N} |a_n - c_n|^2 \underbrace{\|e_n\|^2}_{1} = \sum_{n=-N}^{N} |a_n - c_n|^2 = 0$$

Which implies $a_n = c_n$ for all n $\square$

4. Mean-Square Convergence

Recall: So far we have proven several nice results about convergence of Fourier series, such as:

If f'' is continuous then $S_N(f)$ converges to f uniformly

If f is Lipschitz or differentiable at x then $S_N(f)(x)$ converges to $f(x)$

What if f is just continuous? There is an example showing that $S_N(f)(x)$ might diverge for every x!!![1]

That said we do have the following result:

Theorem: [Mean-Squared Convergence]

If f is continuous and 2π periodic, then $S_N(f)$ converges to f in the mean-squared sense, that is

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}|S_N(f)(x)-f(x)|^2\,dx=0$$

The proof uses the best approximation lemma above, as well as the following:

Theorem: [Approximation]

If f is continuous and periodic with period 2π and $\epsilon>0$ then there is a trigonometric polynomial P such that for all x, we have

$$|P(x)-f(x)|<\epsilon$$

Proof: Just an application of Stone-Weierstrass, where we identify 2π periodic functions with functions on the unit circle in $\mathbb{R}^2$ (compact) and $\mathcal{A}$ is the set of trigonometric polynomials. $\square$

Proof of Mean-Squared Convergence: Let $\epsilon>0$ be given, then by the above there is a trigonometric polynomial P such that for all x

$$|f(x)-P(x)|<\epsilon$$

Notice in particular that

[1] See section 2.2 in Chapter 3 of Stein and Shakarchi

$$\|f - P\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{|f(x) - P(x)|^2}_{<\epsilon^2} dx < \frac{\epsilon}{2\pi}(2\pi) = \epsilon^2$$

Hence $\|f - P\| < \epsilon$

Then for some M and some c_n we have

$$P(x) = \sum_{n=-M}^{M} c_n e_n$$

Define $c_n = 0$ outside of $[-M, M]$ and so by the best approximation Lemma, if $N > M$, we have

$$\|f - S_N(f)\| \leq \left\| f - \sum_{n=-N}^{N} c_n e_n \right\| = \left\| f - \underbrace{\sum_{n=-M}^{M} c_n e_n}_{P} \right\| = \|f - P\| < \epsilon$$

(In the middle, we used the fact that $c_n = 0$ outside of $[-M, M]$)

And therefore $\lim_{N\to\infty} \|f - S_N(f)\| = 0$ $\square$

Note: This result is true even if f is just integrable! In that case it's enough to approximate f with continuous functions.

5. Parseval's Theorem

Video: Parseval's Theorem

Finally, we can deduce a result, which, in my opinion, is the most fun fact about Fourier series!

Theorem: [Parseval's Theorem]

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

Proof: From the orthogonal decomposition, we have

$$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{n=-N}^{N} |a_n|^2$$

$$\sum_{n=-N}^{N} |a_n|^2 = \|f\|^2 - \|f - S_N(f)\|^2$$

Letting $N \to \infty$ in the above, using mean-squared convergence, we get

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \lim_{N\to\infty} \sum_{n=-N}^{N} |a_n|^2 = \|f\|^2 - \underbrace{\lim_{N\to\infty} \|f - S_N(f)\|^2}_{=0} = \|f\|^2 \quad \square$$

Cool Application: Let $f(x) = x$ on $(-\pi, \pi)$. We have shown that the Fourier series of f is

$$\sum_{n\neq 0} \frac{(-1)^{n+1}}{in} e^{inx}$$

Parseval's Theorem then says that

$$
\begin{aligned}
\sum_{n\neq 0}\left|\frac{(-1)^{n+1}}{in}\right|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx \\
\sum_{n\neq 0}\frac{1}{n^2} &= \frac{1}{2\pi}\left(\frac{2\pi^3}{3}\right) \\
\sum_{n=-\infty}^{-1}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{3} \\
2\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{3} \\
\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{1}{2}\left(\frac{\pi^2}{3}\right) = \frac{\pi^2}{6}
\end{aligned}
$$

WOW!!! We can derive other fun identities like those, by considering different functions f.