SOLUTIONS

1. (*10 points*) Using the change of variables below, find the area of the region D bounded by the ellipse $x^2 - xy + y^2 = 8$

$$\begin{cases} x = \sqrt{2}u - \sqrt{\frac{2}{3}}v \\ y = \sqrt{2}u + \sqrt{\frac{2}{3}}v \end{cases}$$

1) $\text{AREA}(D) = \iint_D 1 \, dx\,dy$

2) <u>FIND D'</u>

$x^2 - xy + y^2 = 8 \implies \left(\sqrt{2}u - \sqrt{\frac{2}{3}}v\right)^2 - \left(\sqrt{2}u - \sqrt{\frac{2}{3}}v\right)\left(\sqrt{2}u + \sqrt{\frac{2}{3}}v\right) + \left(\sqrt{2}u + \sqrt{\frac{2}{3}}v\right)^2 = 8$

$\implies \cancel{2u^2} - \cancel{2\frac{2}{\sqrt{3}}uv} + \frac{2}{3}v^2 - \cancel{2u^2} + \frac{2}{3}v^2 + 2u^2 + \cancel{2\frac{2}{\sqrt{3}}uv} + \frac{2}{3}v^2 = 8$

$\implies 2u^2 + \left(\frac{2}{3} + \frac{2}{3} + \frac{2}{3}\right)v^2 = 8$

$\implies 2u^2 + 2v^2 = 8$

$\implies \underline{u^2 + v^2 = 4}$ $\rightsquigarrow$ SO D' IS A DISK OF RADIUS 2

3) $dx\,dy = \left|\frac{dx\,dy}{du\,dv}\right| du\,dv$

(TURN PAGE)

$$\frac{dx\,dy}{dU\,dV} = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{vmatrix} = \begin{vmatrix} \sqrt{2} & -\sqrt{\frac{2}{3}} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{vmatrix}$$

$$= \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

so $\quad dx\,dy = \left|\frac{4}{\sqrt{3}}\right| dU\,dV = \frac{4}{\sqrt{3}}\,dU\,dV$

4) $\quad \text{AREA}(D) = \iint_D 1\,dx\,dy$

$$= \iint_{D'} 1 \left(\frac{4}{\sqrt{3}}\right) dU\,dV$$

$$= \frac{4}{\sqrt{3}} \iint_{D'} 1\,dU\,dV$$

$$= \frac{4}{\sqrt{3}}\,\text{AREA}(D')$$

$D' =$ DISK OF RADIUS 2

$$= \frac{4}{\sqrt{3}}\,\pi(2)^2$$

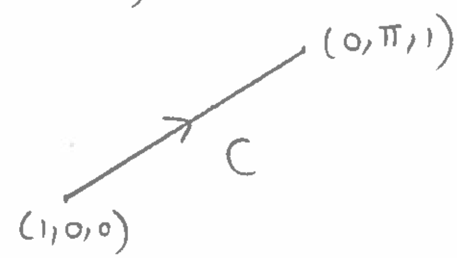
$$= \frac{16}{\sqrt{3}}\pi$$

2. *(10 points)* Find $\int_C F \cdot dr$, where

$$F(x, y, z) = \langle e^x \sin(yz), ze^x \cos(yz), ye^x \cos(yz) \rangle$$

C is the line connecting $(1, 0, 0)$ and $(0, \pi, 1)$.

1) PICTURE (OPTIONAL)



2) F CONSERVATIVE (SEE SECTION 16.5)

3) ANTIDERIVATIVE

$$F = \nabla f \Rightarrow \langle e^x \sin(yz), ze^x \cos(yz), ye^x \cos(yz) \rangle = \langle f_x, f_y, f_z \rangle$$

$$f_x = e^x \sin(yz) \Rightarrow f = \int e^x \sin(yz)\, dx = e^x \sin(yz) + \text{JUNK}$$

$$f_y = ze^x \cos(yz) \Rightarrow f = \int ze^x \cos(yz)\, dy = \cancel{z} e^x \frac{\sin(yz)}{\cancel{z}} = e^x \sin(yz) + \text{JUNK}$$

$$f_z = ye^x \cos(yz) \Rightarrow f = \int ye^x \cos(yz)\, dz = \cancel{y} e^x \frac{\sin(yz)}{\cancel{y}} = e^x \sin(yz) + \text{JUNK}$$

HENCE $f(x, y, z) = e^x \sin(yz)$

4) $$\int_C F \cdot dr = \int_C \nabla f \cdot dr \overset{\text{FTC}}{=} f(0, \pi, 1) - f(1, 0, 0) = e^0 \cancelto{0}{\sin(\pi(1))} - e^1 \cancelto{0}{\sin(00)} = 0$$

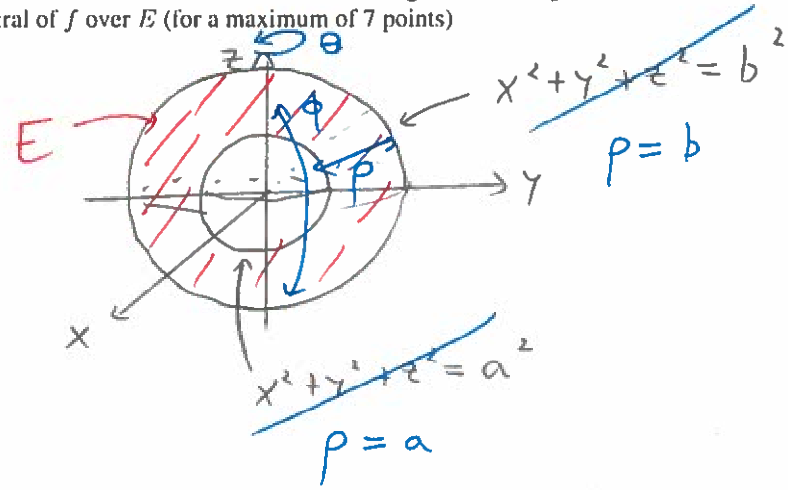
3. *(10 points)* Find the average value of the function

$$f(x,y,z) = \cancel{\frac{1}{\sqrt{x^2+y^2+z^2}}} \quad 7\left(x^2+y^2+z^2\right)^2$$

over the solid E, where E is the region between the two surfaces $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ (Here $b>a>0$). Include a picture of E. ~~No need to simplify your answer.~~

Note: If you don't know how to calculate an average value, then just calculate the triple integral of f over E (for a maximum of 7 points)

1) PICTURE



2) INEQUALITIES

$a \le \rho \le b$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

3) INTEGRATE

$$\text{AVERAGE} = \frac{\iiint_E f(x,y,z)\,dx\,dy\,dz}{\text{VOLUME}(E)}$$

$$= \frac{\int_0^{\pi}\int_0^{2\pi}\int_a^b 7(\rho^2)^2\,\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi}{\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3}$$

← E IS THE DIFFERENCE BETWEEN TWO SPHERES, OR USE $\iiint_E 1$

$$= \frac{7(2)\cancel{\pi} \left(\int_a^b \rho^6 \, d\rho \right) \left(\int_0^\pi \sin(\phi) \, d\phi \right)}{\frac{4}{3}\cancel{\pi} \left(b^3 - a^3 \right)}$$

$$= \frac{\cancel{7}(\cancel{2}) \left(\frac{3}{\cancel{4}} \right) \left[\frac{\rho^7}{\cancel{7}} \right]_a^b \left[-\cos(\phi) \right]_0^\pi}{b^3 - a^3}$$

$$= \cancel{2}$$

$$= 3 \left(\frac{b^7 - a^7}{b^3 - a^3} \right)$$

4. *(10 points)* Calculate

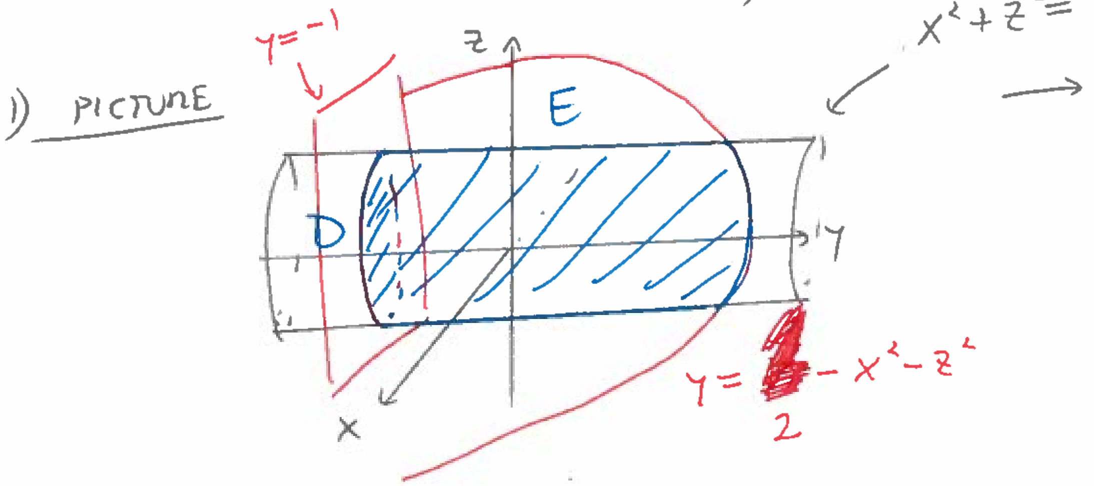
2

$$\iiint_E \text{[redacted]} \, dxdydz$$

E is the solid inside the surface $x^2 + z^2 =$ [redacted] 1 and between the surfaces $y =$
2 [redacted] $- x^2 - z^2$ and $y = -1$

(INCLUDE A PICTURE OF E)

1) PICTURE



2) INEQUALITIES

SMALL $\leq y \leq$ BIG

$-1 \leq y \leq 2 - x^2 - z^2$

$$\boxed{-1 \leq y \leq 2 - r^2}$$

3) FIND D

D = DISK OF RADIUS 1

(BECAUSE OF $x^2 + z^2 = 1$)

$$\boxed{\begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array}}$$

4) INTEGRATE

$$\iiint_E y \, dx\,dy\,dz = \int_0^{2\pi} \int_0^1 \int_{-1}^{2-r^2} 2r \, dy\, dr\, d\theta$$

$$= 2\pi \int_0^1 2r \left(2 - r^2 - (-1)\right) dr$$

$$= 2\pi \int_0^1 2r \left(3 - r^2\right) dr$$

$$= 2\pi \int_0^1 6r - 2r^3 \, dr$$

$$= 2\pi \left[3r^2 - \frac{r^4}{2} \right]_0^1$$

$$= 2\pi \left(3 - \frac{1}{2}\right)$$

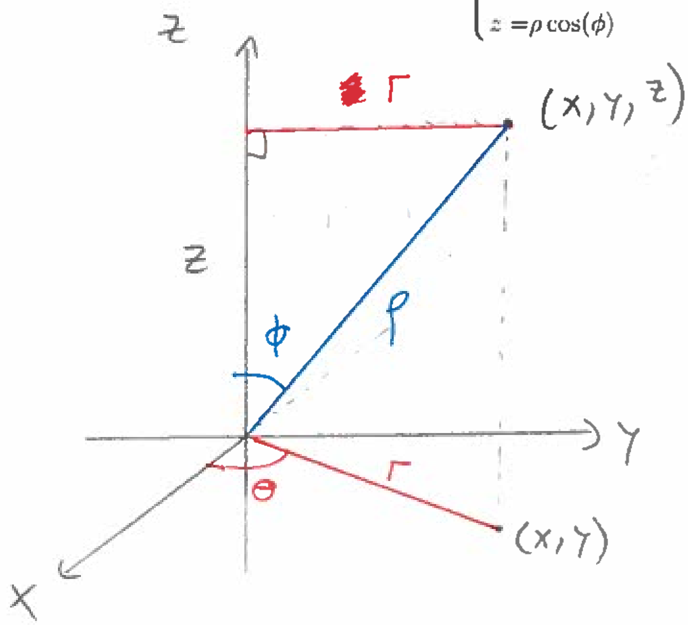
$$= 2\pi \left(\frac{5}{2}\right)$$

$$= 5\pi$$

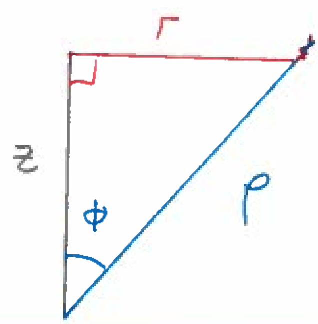
5. *(10 points)* Derive (from scratch) the equations for spherical coordinates: (INCLUDE PICTURES)

$$\begin{cases} x = \rho\sin(\phi)\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases}$$

1) PICTURE



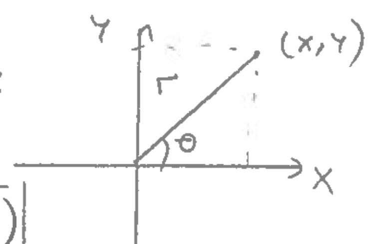
2) FIRST FOCUS ON THE TRIANGLE



THEN $\cos(\phi) = \dfrac{z}{\rho} \implies \boxed{z = \rho\cos(\phi)}$

MOREOVER, $\sin(\phi) = \dfrac{r}{\rho} \implies r = \rho\sin(\phi)$

3) FINALLY, BY POLAR COORDINATES:



$$\boxed{\begin{aligned} x &= r\cos(\theta) = \rho\sin(\phi)\cos(\theta) \\ y &= r\sin(\theta) = \rho\sin(\phi)\sin(\theta) \end{aligned}}$$