SOLUTIONS

MATH 2D – MAKE–UP FINAL

Name: ______________________

Student ID: ______________________

Instructions: This is it, your final hurdle to freedom!!! You have 120 minutes to take this exam, for a total of 100 points. This is a closed book and closed notes exam and calculators and/or portable electronic devices are not allowed. Remember that you are not only graded on your final answer, but also on your work. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. May the Chen Lou be with you!!!

Academic Honesty Statement: I hereby certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating will be subject to disciplinary consequences, pursuant to section 102.1 of the UCI Student Code of Conduct.

Signature: ______________________________

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Date: Tuesday, June 12, 2018.

1. *(10 points)* Estimate the error in calculating the volume of the box if the box has dimensions $x = 10$, $y = 20$, $z = 30$, and the errors in calculating the sides are $dx = -0.1$, $dy = 0.2$, $dz = 0.3$.

$$V = xyz$$

$$\Delta V \approx dV = V_x\, dx + V_y\, dy + V_z\, dz$$

ACTUAL ERROR

$$= yz\, dx + xz\, dy + xy\, dz$$

$$= (20)(30)(-0.1) + (10)(30)(0.2) + (10)(20)(0.3)$$

$$= \cancel{-60} \cancel{+60} + 60$$

$$= 60$$

2. *(10 points)* Show that $u(x, y) = \ln(x^2 + y^2)$ solves Laplace's equation

$$u_{xx} + u_{yy} = 0$$

1) $U_x = (\ln(x^2+y^2))_x = \frac{1}{x^2+y^2}(2x) = \frac{2x}{x^2+y^2}$

$U_{xx} = \frac{2(x^2+y^2) - (2x)(2x)}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$

2) $U_y = \frac{1}{x^2+y^2} 2y = \frac{2y}{x^2+y^2}$

$U_{yy} = \frac{2(x^2+y^2) - (2y)(2y)}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$

3) $U_{xx} + U_{yy} = \frac{\cancel{2y^2} - \cancel{2x^2}}{(x^2+y^2)^2} + \frac{\cancel{2x^2} - \cancel{2y^2}}{(x^2+y^2)^2} = 0$ ✓

3. *(10 points)* The two sub-parts of this problem are independent of each other

   (a) *(5 points)* Suppose that for all t, $\mathbf{u}(t)$ is a unit vector. Show that $\mathbf{u}'(t)$ is always perpendicular for $\mathbf{u}(t)$.

   Hint: $\|\mathbf{u}(t)\|^2 = \mathbf{u}(t) \cdot \mathbf{u}(t)$.

BY THE HINT $\underbrace{\|u(t)\|^2}_{1} = u(t) \cdot u(t)$ (SINCE U IS A UNIT VECTOR)

$\Rightarrow \quad u(t) \cdot u(t) = 1$

DIFFERENTIATE

$\Rightarrow \quad (u(t) \cdot u(t))' = (1)'$

$\Rightarrow \quad u'(t) \cdot u(t) + u(t) \cdot u'(t) = 0$

$\Rightarrow \quad 2u'(t) \cdot u(t) = 0 \quad \Rightarrow \boxed{u'(t) \cdot u(t) = 0}$ (SO $u'(t)$ IS PERPENDICULAR TO $u(t)$)

   (b) *(5 points)* Suppose $\mathbf{u}$ is a unit vector. Find the (vector) projection of ∇f on $\mathbf{u}$ and express your answer in terms of $D_{\mathbf{u}} f$ (the directional derivative of f in the direction $\mathbf{u}$). Simplify your answer as much as possible.

$\mathrm{proj}_u \nabla f = \left(\dfrac{\nabla f \cdot u}{u \cdot u}\right) u$

$u \cdot u = \|u\|^2 = 1$ SINCE U IS A UNIT VECTOR

$= (\nabla f \cdot u)\, u$

$= (D_u f)\, u$

4. *(10 points)* Find the equation of the tangent plane to the surface $x^4 + y^4 + z^4 = 3x^2y^2z^2$ at the point $(1, 1, 1)$. Write your answer in the form $ax + by + z = c$ for some a, b, c.

1) $F(x,y,z) = x^4 + y^4 + z^4 - 3x^2y^2z^2$

2) EQUATION $F_x(1,1,1)(x-1) + F_y(1,1,1)(y-1) + F_z(1,1,1)(z-1) = 0$

$F_x = 4x^3 - 6xy^2z^2 \Rightarrow 4 - 6 = -2$

AT $(1,1,1)$

$F_y = 4y^3 - 6x^2yz \Rightarrow 4 - 6 = -2$

$F_z = 4z^3 - 6x^2y^2z \Rightarrow 4 - 6 = -2$

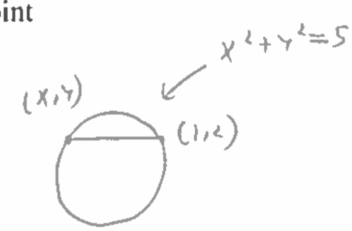
3) EQUATION $(\cancel{-2})(x-1) + (\cancel{-2})(y-1) + (\cancel{-2})(z-1) = 0$

$x - 1 + y - 1 + z - 1 = 0$

ANS $\boxed{x + y + z = 3}$

USE LAGRANGE MULTIPLIERS TO FIND

5. *(10 points)* Find the smallest and largest distance between the point $(1,2)$ and the circle $x^2+y^2=5$.



1) DISTANCE $\quad d=\sqrt{(x-1)^2+(y-2)^2}$

$$\Rightarrow \quad f(x,y)=(x-1)^2+(y-2)^2$$

2) CONSTRAINT $\quad g(x,y)=x^2+y^2-5 \qquad (\text{SINCE } x^2+y^2=5)$

3) LAGRANGE $\quad \begin{cases} f_x=\lambda g_x \\ f_y=\lambda g_y \end{cases} \Rightarrow \begin{cases} \cancel{2}(x-1)=\cancel{2}\lambda x \\ \cancel{2}(y-2)=\cancel{2}\lambda y \end{cases}$

$$\Rightarrow \begin{cases} x(1-\lambda)=1 \\ y(1-\lambda)=2 \end{cases} \Rightarrow \begin{cases} x=\dfrac{1}{1-\lambda} \\ y=\dfrac{2}{1-\lambda} \end{cases}$$

CONSTRAINT $\quad x^2+y^2=5$

$$\left(\frac{1}{1-\lambda}\right)^2+\left(\frac{2}{1-\lambda}\right)^2=5$$

$$\frac{\cancel{5}}{(1-\lambda)^2}=\cancel{5}$$

$$\frac{1}{(1-\lambda)^2}=1 \Rightarrow (1-\lambda)^2=1 \Rightarrow 1-\lambda=1 \text{ OR } 1-\lambda=-1$$

$$\Rightarrow \underline{\lambda=0} \text{ OR } \underline{\lambda=2}$$

CASE 1 $\quad \underline{\lambda=0} \Rightarrow x=\dfrac{1}{1-0}=1, \quad y=\dfrac{2}{1-0}=2 \Rightarrow (1,2)$

CASE 2 $\quad \underline{\lambda=2} \Rightarrow x=\dfrac{1}{1-2}=-1, \quad y=\dfrac{2}{1-2}=-2 \Rightarrow (-1,-2)$

COMPARE $\quad f(1,2)=0 \longrightarrow$ ABS MIN

$f(-1,-2)=2^2+4^2=20 \longrightarrow$ ABS MAX

4) ANS SMALLEST 0, LARGEST $\sqrt{20}$

6. *(10 points)*

(a) *(8 points)* Use **Lagrange multipliers** to show that the triangle with maximum area that has a given perimeter p must be equilateral.

Hint: The area of a triangle with sides x, y, z is $A = \sqrt{s(s-x)(s-y)(s-z)}$ where $s = \frac{p}{2}$. Assume that $s \neq x, y, z$.

1) $f(x,y,z) = s(s-x)(s-y)(s-z)$ (SQUARE A)

$g(x,y,z) = x+y+z-p$ (SINCE $x+y+z=p$)

2) $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \end{cases} \Rightarrow \begin{cases} s(-1)(s-y)(s-z) = \lambda \quad ① \\ s(s-x)(-1)(s-z) = \lambda \quad ② \\ s(s-x)(s-y)(-1) = \lambda \end{cases}$

3) ① $\Rightarrow -s(s-y)(\cancel{s-z}) = -s(s-x)(\cancel{s-z}) \Rightarrow \cancel{s}-y = \cancel{s}-x \Rightarrow y = x$

② $\Rightarrow -s(\cancel{s-x})(s-z) = -s(\cancel{s-x})(s-y) \Rightarrow \cancel{s}-z = \cancel{s}-y \Rightarrow z = y$

4) HENCE $x = y = z$ AND THE TRIANGLE IS EQUILATERAL

(b) *(2 points)* What kind of triangle do you get if $s = x$?

IF $s = x$ THEN $\frac{p}{2} = x \Rightarrow p = 2x$

$\Rightarrow x + y + z = 2x$

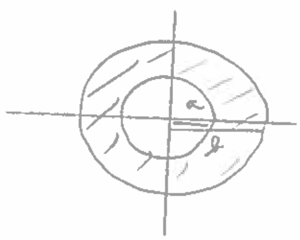
$\Rightarrow x = y + z$

SO ONE SIDE OF THE TRIANGLE IS EQUAL TO THE SUM OF ITS TWO OTHER SIDES

$\Rightarrow$ THE TRIANGLE IS FLAT!

7. *(10 points)* Find the average value of the function $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$ on the ring $a^2 \leq x^2 + y^2 \leq b^2$ where $b > a > 0$. Simplify your answer as much as possible

1) $$\text{AVG} = \frac{\iint_D f(x,y)\,dx\,dy}{\text{AREA}(D)}$$

D = 

2) $$\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{2\pi}\int_a^b \overset{\cancel{\frac{1}{r}}}{\overset{\|}{\frac{1}{\sqrt{r^2}}}}\,\cancel{r}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_a^b 1\,dr\,d\theta$$

$$= \int_0^{2\pi} (b-a)\,d\theta = (2\pi)(b-a)$$

3) $\text{AREA}(D) = \pi b^2 - \pi a^2$ (= OUTSIDE DISK − INSIDE DISK),

(or $\iint_D 1\,dx\,dy$)

4) ANS $$\frac{2\cancel{\pi}(b-a)}{\cancel{\pi}(b^2-a^2)} = \frac{2(b-a)}{b^2-a^2} = \frac{2\cancel{(b-a)}}{\cancel{(b-a)}(b+a)} = \frac{2}{b+a}$$

8. *(10 points)* Write $\int_0^{2}\int_0^{4-x^2}\int_0^{1-z/4} f(x,y,z)\,dy\,dz\,dx$ as

$$\int_?^?\int_?^?\int_?^? f(x,y,z)\,dx\,dz\,dy$$

Illustrate with pictures.

1) KNOW

$0 \le y \le 1-\frac{z}{4}$

$0 \le z \le 4-x^2$ (NO Y)

$0 \le x \le 2$ (CONSTANT)

WANT

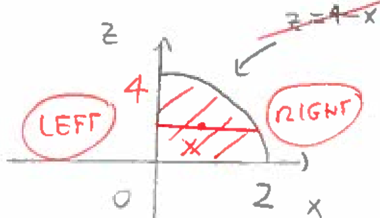
$? \le x \le ?$

$? \le z \le ?$ (NO X)

$? \le y \le ?$ (CONST.)

2) PICTURES

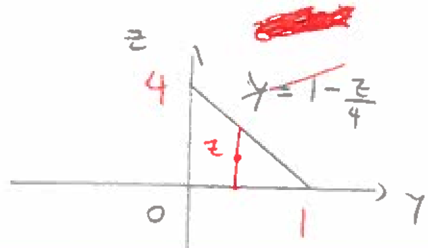
$\cancel{z=4-x^2} \Rightarrow x=\sqrt{4-z}$



$0 \le x \le 2$

$0 \le z \le 4-x^2$

$z = 4-4y$



$\cancel{y=1-\frac{z}{4}}$

$0 \le y \le 1-\frac{z}{4}$

NEW $0 \le z \le 4$

3) LEFT $\le x \le$ RIGHT

$0 \le x \le \sqrt{4-z}$

SMALLER $\le z \le$ BIGGER

$0 \le z \le 4-4y$ (NO x ✓)

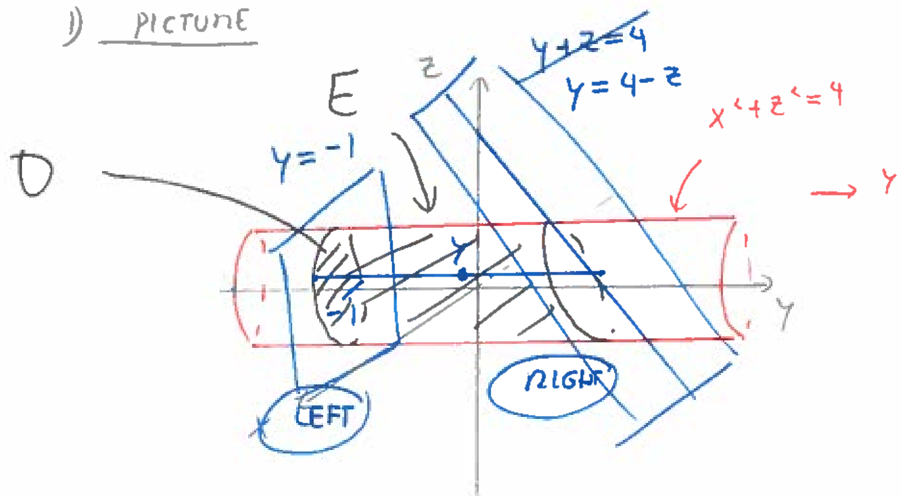
$0 \le y \le 1$ (CONST ✓)

4) ANS

$$= \boxed{\int_0^1\int_0^{4-4y}\int_0^{\sqrt{4-z}} f(x,y,z)\,dx\,dz\,dy}$$

9. *(10 points)* Find the volume of the solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$

1) PICTURE



2) $V = \iiint_E 1 \, dx\,dy\,dz$

LEFT $\leq y \leq$ RIGHT

$-1 \leq y \leq 4 - z$

so $V = \iint_D \int_{-1}^{4-z} 1 \, dy \, dx\,dz = \iint_D 4 - z + 1 \, dx\,dz = \iint_D 5 - z \, dx\,dz$

3) D = SHADOW TO THE LEFT OF E

D = DISK $x^2 + z^2 = 4$ (INTERSECTION OF $x^2 + z^2 = 4$ AND $y = -1$)

2 D

so $V = \int_0^{2\pi} \int_0^2 (5 - r \sin(\theta)) r \, dr\, d\theta = \int_0^{2\pi} \int_0^2 5r - r^2 \sin(\theta) \, dr\, d\theta$

$= \int_0^{2\pi} \left[\frac{5}{2} r^2 - \frac{r^3}{3} \sin(\theta) \right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} 10 - \frac{8}{3} \sin(\theta) \, d\theta$

$= \left[10\theta + \frac{8}{3} \cos(\theta) \right]_0^{2\pi} = (10)(2\pi) = 20\pi$

10. *(10 points)* The Grand Finale!!! Assume $b > a > 0$. Calculate the integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$$

Hint: Calculate the volume under the function $z = e^{-xy}$ and over the rectangle $[0, \infty) \times [a, b]$ in two different ways.

1) ON THE ONE HAND, THE VOLUME IS GIVEN BY

$$\int_0^\infty \int_a^b e^{-xy}\,dy\,dx = \int_0^\infty \left[\frac{e^{-xy}}{-x}\right]_{y=a}^{y=b} dx$$

$$= \int_0^\infty \frac{e^{-xb}}{-x} + \frac{e^{-xa}}{x}\,dx$$

$$= \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$$

2) ON THE OTHER HAND, BY FUBINI, THE VOLUME IS GIVEN BY

$$\int_a^b \int_0^\infty e^{-xy}\,dx\,dy = \int_a^b \left[\frac{e^{-xy}}{-y}\right]_{x=0}^{x=\infty} dy$$

0 SINCE $Y \geq a > 0$ 1

$$= \int_a^b \frac{e^{-\infty y}}{-y} + \frac{e^{-0y}}{y}\,dy$$

$$= \int_a^b \frac{1}{y}\,dy = LN(b) - LN(a)$$

3) SINCE THE TWO INTEGRALS ARE EQUAL (BY FUBINI), WE GET:

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \boxed{LN(b) - LN(a)}$$