MATH 409 – MIDTERM 1 – SOLUTIONS

1. **Option 1: The Monotone Sequence Theorem**

   Suppose (s_n) is increasing and bounded above (by M)

   Consider $S = \{s_n \mid n \in \mathbb{N}\}$

   Then S is bounded above (by M) and therefore has a least upper bound s.

   Claim: (s_n) converges to s

   Proof: Let $\epsilon > 0$ be given, and consider $s - \epsilon < s = \sup(S)$, and hence there is s_N such that $s_N > s - \epsilon$.

   With that N, if $n > N$, since (s_n) is increasing we have $s_n > s_N > s - \epsilon$, so $s_n > s - \epsilon$, so $s_n - s > -\epsilon$.

   But also by definition of sup, $s_n \leq s$, so $s_n - s \leq 0 < \epsilon$.

   And putting both things together, we get $-\epsilon < s_n - s < \epsilon$, so $|s_n - s| < \epsilon$ ✓ □

Date: Thursday, September 30, 2021.

Option 2: If (s_n) converges to s, then (s_n) is bounded

Suppose $s_n \to s$, then by the definition of a limit with $\epsilon = 1$, we get that there is N such that if $n > N$, then $|s_n - s| < 1$.

With that N, if $n > N$, then

$$|s_n| = |s_n - s + s| \leq |s_n - s| + |s| < 1 + |s|$$

Let $M = \max\{|s_1|, \cdots, |s_N|, 1 + |s|\}$.

Case 1: $n \leq N$, then

$$|s_n| \leq \max\{|s_1|, \cdots, |s_N|\} \leq M$$

Case 2: $n > N$, then

$$|s_n| < 1 + |s| \leq M$$

So in either case $|s_n| \leq M$ ✓ □

2. (a) For all $\epsilon > 0$ there is N such that if $n > N$, then $|s_n - s| < \epsilon$

(b) **STEP 1:** Scratchwork

$$\left|\frac{2n+3}{5n+7} - \frac{2}{5}\right| = \left|\frac{(2n+3)(5) - 2(5n+7)}{5(5n+7)}\right| = \left|\frac{10n+15-10n-14}{5(5n+7)}\right| = \frac{1}{5(5n+7)} < \epsilon$$

Which gives:

$$\frac{1}{5n+7} < 5\epsilon \Rightarrow 5n+7 > \frac{1}{5\epsilon} \Rightarrow 5n > \frac{1}{5\epsilon} - 7 \Rightarrow n > \frac{1}{25\epsilon} - \frac{7}{5}$$

Which suggests to let $N = \frac{1}{25\epsilon} - \frac{7}{5}$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $N = \frac{1}{25\epsilon} - \frac{7}{5}$, then if $n > N$, we get:

$$\left|\frac{2n+3}{5n+7} - \frac{2}{5}\right| = \frac{1}{5(5n+7)}$$

But if $n > N = \frac{1}{25\epsilon} - \frac{7}{5}$, then $5n > \frac{1}{5\epsilon} - 7$ so $5n + 7 > \frac{1}{5\epsilon}$ so $\frac{1}{5n+7} < 5\epsilon$ and so $\frac{1}{5(5n+7)} < \epsilon$, hence

$$\left|\frac{2n+3}{5n+7} - \frac{2}{5}\right| = \frac{1}{5(5n+7)} < \epsilon$$

And therefore $\lim_{n\to\infty} \frac{2n+3}{5n+7} = \frac{2}{5}$ $\square$

3. (a) If (s_n) is a sequence that is increasing and bounded above, then (s_n) converges.

(b) **Claim # 1:** $s_n \geq \sqrt{2}$ for all n

Proof: Let P_n be the proposition: $s_n \geq \sqrt{2}$

Base Case: $s_1 = 2 = \sqrt{4} \geq \sqrt{2}$ ✓

Inductive Step: Suppose P_n is true, that is $s_n \geq \sqrt{2}$, show P_{n+1} is true, that is $s_{n+1} \geq \sqrt{2}$, but:

$$
\begin{aligned}
s_{n+1} - \sqrt{2} &= \left(\frac{s_n}{2} + \frac{1}{s_n}\right) - \sqrt{2} \\
&= \frac{(s_n)^2 + 2 - 2\sqrt{2}s_n}{2s_n} \\
&= \frac{(s_n)^2 - 2\sqrt{2}s_n + \left(\sqrt{2}\right)^2}{2s_n} \\
&= \frac{\left(s_n - \sqrt{2}\right)^2}{2s_n} \geq 0
\end{aligned}
$$

Hence $s_{n+1} \geq \sqrt{2}$, and by induction $s_n \geq 0$ for all n

Claim # 2: s_n is decreasing

$$
s_{n+1} - s_n = \frac{s_n}{2} + \frac{1}{s_n} - s_n = -\frac{s_n}{2} + \frac{1}{s_n} = \frac{-(s_n)^2 + 2}{2s_n}
$$

However, since $s_n \geq \sqrt{2}$, $(s_n)^2 \geq 2$, so $2 - (s_n)^2 \leq 0$ and so $s_{n+1} - s_n \leq 0$ so $s_{n+1} \leq s_n$

Finally, since (s_n) is decreasing and bounded below (by M), (s_n) converges, so taking $n \to \infty$ in $s_{n+1} = \frac{s_n}{2} + \frac{1}{s_n}$, we get

$$\begin{aligned} s &= \frac{s}{2} + \frac{1}{s} \\ s - \frac{s}{2} &= \frac{1}{s} \\ \frac{s}{2} &= \frac{1}{s} \\ s^2 &= 2 \\ s &= \sqrt{2} \end{aligned}$$

(Here we use the fact that $s_n \geq 0$ so $s \geq 0$)

Hence s_n converges to $\sqrt{2}$

4. (a) If S is bounded above by M, then $\sup(S) = M$ means that for all $M_1 < M$ there is $s \in S$ such that $s > M_1$.

(b) Let $M = \sup(S)$ and let's show that $\sup(kS) = kM$

Upper Bound: Suppose $x \in kS$, then $x = ks$ for some $s \in S$, but since M is an upper bound for S and $k > 0$, we have $x = ks \leq kM$, so kM is an upper bound for kS ✓

Least Upper Bound: Suppose $M_1 < kM$, then $\frac{M_1}{k} < M = \sup(S)$, so by definition of sup there is $s \in S$ with $s > \frac{M_1}{k}$, but then $ks > M_1$, so if you let $x = ks \in kS$, then $x > M_1$ ✓

Hence $\sup(kS) = kM$. □

(c) The statement is **FALSE** if $k < 0$. For example, let $S = [1, 2]$ and $k = -3$, then $kS = -3S = [-6, -3]$, so $\sup(kS) = -3$, but $k \sup(S) = -3 \sup(S) = -3(2) = -6 \Rightarrow\Leftarrow$

You could also have chosen $S = \{0, 1\}$ for example, that would have worked too.

5. (a)

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n \mid n > N\}$$

(b) Notice that for all N, if $n > N$, then we have:

$$s_n \leq \sup\{s_n \mid n > N\}$$
$$t_n \leq \sup\{t_n \mid n > N\}$$

Therefore

$$s_n + t_n \leq \sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\}$$

Taking the sup of the left hand side over $n > N$, we get:

$$\sup\{s_n + t_n \mid n > N\} \leq \sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\}$$

Finally, taking the limit as $N \to \infty$ on both sides, we get:

$$\begin{aligned}
\limsup_{n\to\infty} s_n + t_n &= \lim_{N\to\infty} \sup\{s_n + t_n \mid n > N\} \\
&\leq \lim_{N\to\infty} \sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\} \\
&= \lim_{N\to\infty} \sup\{s_n \mid n > N\} + \lim_{N\to\infty} \sup\{t_n \mid n > N\} \\
&= \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n
\end{aligned}$$

(c) Let $s_n = (-1)^n$ and $t_n = -s_n = -(-1)^n = (-1)^{n+1}$

Then $s_n + t_n = (-1)^n - (-1)^n = 0$, so

$$\limsup_{n\to\infty} s_n + t_n = \limsup_{n\to\infty} 0 = 0$$

But $\limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n = \limsup_{n\to\infty} (-1)^n + \limsup_{n\to\infty} (-1)^{n+1} = 1 + 1 = 2$

$$\text{Hence: } \limsup_{n\to\infty} (s_n + t_n) \neq \left(\limsup_{n\to\infty} s_n\right) + \left(\limsup_{n\to\infty} t_n\right)$$