MAT 267 – MIDTERM 1 – SOLUTIONS

1. Multiple Choice

(1) $\boxed{\mathbf{A}}$ A unit vector parallel to $\mathbf{u} = \langle 2, 2, -1 \rangle$ is

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle 2, 2, -1 \rangle}{\sqrt{4+4+1}} = \frac{\langle 2, 2, -1 \rangle}{\sqrt{9}} = \frac{1}{3}\langle 2, 2, -1 \rangle$$

So to obtain a vector of magnitude 2, multiply this by 2 and get

$$\frac{2}{3}\langle 2, 2, -1 \rangle$$

(2) $\boxed{\mathbf{B}}$ Use the Angle Formula

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \|u\| \, \|v\| \cos(\theta) \\
\langle 3, 1 \rangle \cdot \langle 1, 2 \rangle &= \sqrt{9+1}\sqrt{5}\cos(\theta) \\
5 &= \sqrt{10}\sqrt{5}\cos(\theta) \\
\cos(\theta) &= \frac{5}{\sqrt{10}\sqrt{5}} \\
\cos(\theta) &= \frac{\sqrt{5}}{\sqrt{10}} \\
\cos(\theta) &= \sqrt{\frac{5}{10}} \\
\cos(\theta) &= \frac{1}{\sqrt{2}} \\
\theta &= 45^\circ
\end{aligned}$$

(3) $\boxed{\mathbf{B}}$ $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, see Lecture Notes or Book.

(4) $\boxed{\mathbf{C}}$ First of all, parametrize the circle $x^2 + y^2 = 16$ as

$$x(t) = 4\cos(t)$$
$$y(t) = 4\sin(t)$$

And then

$$z(t) = 2y(t) = 2(4\sin(t)) = 8\sin(t)$$

And therefore $\langle 4\cos(t), 4\sin(t), 8\sin(t)\rangle$

(5) $\boxed{\mathbf{D}}$ **Point:** $(2, 4, -5)$

Direction Vector: $\langle 1-2, 2-4, 1-(-5)\rangle = \langle -1, -2, 6\rangle$

Answer:

$\mathbf{r}(t) = \langle 2, 4, -5\rangle + t\langle -1, -2, 6\rangle = \langle 2-t, 4-2t, -5+6t\rangle$

(6) $\boxed{\mathbf{A}}$

$$\mathbf{r}'(t) = \left\langle 2, 6t+4, -\frac{1}{t^2}\right\rangle$$
$$\mathbf{r}'(1) = \langle 2, 10, -1\rangle$$

(7) $\boxed{\mathbf{D}}$

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{k} \\ &= 2\mathbf{i} - (4-1)\mathbf{j} - \mathbf{k} \\ &= \langle 2, -3, -1\rangle \end{aligned}$$

Hence the area is:

$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{4+9+1} = \sqrt{14}$$

(8) $\boxed{\mathbf{C}}$

$$\begin{aligned}\mathbf{r}(t) &= \left\langle \cos(2t), 3t^2 + t^3, e^{-3t}\right\rangle \\ \mathbf{v}(t) &= \mathbf{r}'(t) = \left\langle -2\sin(2t), 6t + 3t^2, -3e^{-3t}\right\rangle \\ \mathbf{a}(t) &= \mathbf{v}'(t) = \left\langle -4\cos(2t), 6 + 6t, 9e^{-3t}\right\rangle \\ \mathbf{a}(0) &= \left\langle -4\cos(0), 6 + 6(0), 9e^{0}\right\rangle = \langle -4, 6, 9\rangle \end{aligned}$$

2. Free Response

1(a)

$$\begin{aligned}
\text{Length} &= \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt \\
&= \int_0^{2\pi} \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2 + 4^2} dt \\
&= \int_0^{2\pi} \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 16} dt \\
&= \int_0^{2\pi} \sqrt{4+16} dt \\
&= \int_0^{2\pi} \sqrt{20} dt \\
&= \sqrt{20}(2\pi - 0) \\
&= 2\pi\sqrt{20}
\end{aligned}$$

(Which can be simplified to $4\pi\sqrt{5}$ if you want)

1(b) **Point:**

$$\begin{aligned}
\mathbf{r}\left(\frac{\pi}{2}\right) &= \left\langle \cos\left(2\left(\frac{\pi}{2}\right)\right), \sin\left(2\left(\frac{\pi}{2}\right)\right), 4\left(\frac{\pi}{2}\right) + 3 \right\rangle \\
&= \langle \cos(\pi), \sin(\pi), 2\pi + 3 \rangle \\
&= \langle -1, 0, 2\pi + 3 \rangle
\end{aligned}$$

Direction Vector:

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -2\sin(2t), 2\cos(2t), 4 \rangle \\
\mathbf{r}'\left(\frac{\pi}{2}\right) &= \left\langle -2\sin\left(2\left(\frac{\pi}{2}\right)\right), 2\cos\left(2\left(\frac{\pi}{2}\right)\right), 4 \right\rangle \\
&= \langle -2\sin(\pi), 2\cos(\pi), 4 \rangle \\
&= \langle 0, -2, 4 \rangle
\end{aligned}$$

Answer:

$$\mathbf{L}(t) = \langle -1, 0, 2\pi + 3\rangle + t\,\langle 0, -2, 4\rangle = \langle -1, -2t, 2\pi + 3 + 4t\rangle$$

2(a)

$$\begin{aligned}\mathbf{a} &= \langle 1-2, 4-1, -1-1\rangle = \langle -1, 3, -2\rangle \\ \mathbf{b} &= \langle 3-2, 2-1, 0-1\rangle = \langle 1, 1, -1\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{n} =& \mathbf{a}\times\mathbf{b} \\ =& \begin{vmatrix} i & j & k \\ -1 & 3 & -2 \\ 1 & 1 & -1 \end{vmatrix} \\ =& \begin{vmatrix} 3 & -2 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -2 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix}\mathbf{k} \\ =& (-3+2)\mathbf{i} - (1+2)\mathbf{j} + (-1-3)\mathbf{k} \\ =& \langle -1, -3, -4\rangle\end{aligned}$$

Since we want the normal vector to start with 1, just let

$$\mathbf{n} = \langle 1, 3, 4\rangle\,, \text{ so } b = 3 \text{ and } c = 4$$

2(b) **Point:** $(2, 1, 1)$

Normal Vector: $\mathbf{n} = \langle 1, 3, 4\rangle$ from part (a)

Equation:

$$\begin{aligned}1(x-2) + 3(y-1) + 4(z-1) =& 0 \\ x - 2 + 3y - 3 + 4z - 4 =& 0 \\ x + 3y + 4z =& 2 + 3 + 4 \\ x + 3y + 4z =& 9\end{aligned}$$

3(a)

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)dt \\ &= \left\langle \int 2t - 3dt, \int 4\sin(t)dt, \int 6e^{-2t}dt \right\rangle \\ &= \left\langle t^2 - 3t + A, -4\cos(t) + B, \frac{6}{-2}e^{-2t} + C \right\rangle \\ &= \left\langle t^2 - 3t + A, -4\cos(t) + B, -3e^{-2t} + C \right\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{r}(0) &= \left\langle 0^2 - 3(0) + A, -4\cos(0) + B, -3e^{-2(0)} + C \right\rangle \\ &= \langle A, -4 + B, -3 + C \rangle \\ &= \langle 2, 4, 3 \rangle\end{aligned}$$

Hence $A = 2$, $-4 + B = 4 \Rightarrow B = 8$ and $-3 + C = 3 \Rightarrow C = 6$ and hence

$$\mathbf{r}(t) = \left\langle t^2 - 3t + 2, -4\cos(t) + 8, -3e^{-2t} + 6 \right\rangle$$

3(b)

$$\mathbf{a}(t) = \mathbf{v}'(t) = \left\langle 2, 4\cos(t), -12e^{-2t} \right\rangle$$

By Newton's Second Law:

$$\begin{aligned}\mathbf{F}(t) &= m\mathbf{a}(t) \\ &= 5\left\langle 2, 4\cos(t), -12e^{-2t} \right\rangle \\ &= \left\langle 10, 20\cos(t), -60e^{-2t} \right\rangle\end{aligned}$$

Hence

$$\mathbf{F}(0) = \left\langle 10, 20\cos(0), -60e^{-2(0)} \right\rangle = \langle 10, 20, -60\rangle$$

And

$$\begin{aligned} \|\mathbf{F}(0)\| &= \|\langle 10, 20, -60\rangle\| \\ &= \sqrt{10^2 + 20^2 + (-60)^2} \\ &= \sqrt{100 + 400 + 3600} \\ &= \sqrt{4100} \end{aligned}$$

(Which you can write as $10\sqrt{41}N$ if you want)