SOLUTIONS

1. *(10 points)* Find the **vector** equation of the tangent line to the following curve at (5, 2, 8)

$$\mathbf{r}(t) = \langle 1 + 2t, t^3 - 3t, 2t^2 \rangle$$

1) FIND t

$\langle 1+2t, t^3-3t, 2t^2 \rangle = \langle 5, 2, 8 \rangle$

$1+2t = 5 \Rightarrow 2t = 4 \Rightarrow t = 2$

CHECK $t^3 - 3t = 2^3 - 3(2) = 8-6=2$ ✓

$2t^2 = 2(2^2) = 8$ ✓

HENCE $t = 2$

2) POINT $(5, 2, 8)$

3) DIRECTION VECTOR

$r'(t) = \langle 2, 3t^2 - 3, 4t \rangle$

$r'(2) = \langle 2, 3(2)^2 - 3, 4(2) \rangle = \langle 2, 9, 8 \rangle$

4) ANSWER $L(t) = \langle 5, 2, 8 \rangle + \langle 2, 9, 8 \rangle t$

Answer	$L(t) = \langle 5+2t, 2+9t, 8+8t \rangle$

2. *(10 points)* For the following surface, find the:

(1) Name of the surface

(2) Center of the surface

(3) Direction the surface is facing in (x, y, z direction)

(4) A *very* rough sketch of the surface, just to get an idea of the general shape

$$x^2 - y^2 + z^2 = 2x + 8y - 6z - 32$$

$$x^2 - 2x - y^2 - 8y + z^2 + 6z = -32$$

$$(x-1)^2 - 1^2 - (y^2 + 8y) + (z+3)^2 - 3^2 = -32$$

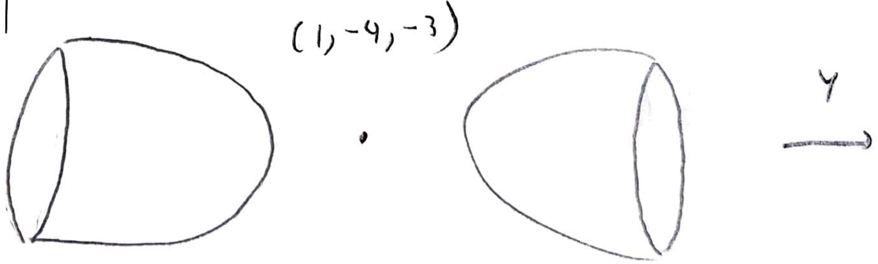
$$(x-1)^2 - [(y+4)^2 - 4^2] + (z+3)^2 = -32 + 1 + 9$$

Name: HYPERBOLOID OF TWO SHEETS (TWO CUPS)

Center: $(1, -4, -3)$

Direction: Y-DIRECTION

Sketch:



$$(x-1)^2 - (y+4)^2 + 16 + (z+3)^2 = -22$$

$$(x-1)^2 - (y+4)^2 + (z+3)^2 = -22 - 16 = -38$$

$$-(x-1)^2 + (y+4)^2 - (z+3)^2 = 38$$

3. *(10 points)* Find the arclength of the following curve from $t = 0$ to $t = \pi$

$$\mathbf{r}(t) = \left\langle 5\cos(2t), 2 - \left(\sqrt{3}\right)t, 5\sin(2t)\right\rangle$$

$$\text{LENGTH} = \int_0^{\pi} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

$$= \int_0^{\pi} \sqrt{(-5\sin(2t)(2))^2 + (-\sqrt{3})^2 + (5\cos(2t)(2))^2}\, dt$$

$$= \int_0^{\pi} \sqrt{100\sin^2(2t) + 3 + 100\cos^2(2t)}\, dt$$

$$= \int_0^{\pi} \sqrt{103}\, dt$$

$$= \sqrt{103}\,(\pi - 0)$$

$$= \left(\sqrt{103}\right)\pi$$

Arclength:	$\left(\sqrt{103}\right)\pi$

4. *(10 points)* Find the equation of the plane containing the point $(0, -6, 1)$ and the line parametrized by $\mathbf{r}(t) = \langle 2t - 4, t - 3, t \rangle$.

Note: Write your final answer in the form $x = ay + bz + c$ for some a, b, and c.

1) POINTS

$A = (0,-6,1)$, $B = r(0) = (-4,-3,0)$

$C = r(1) = (2-4, 1-3, 1) = (-2,-2,1)$

2) VECTORS

$\underline{a} = \overrightarrow{AB} = \langle -4-0, -3+6, 0-1 \rangle = \langle -4, 3, -1 \rangle$

$\underline{b} = \overrightarrow{AC} = \langle -2-0, -2+6, 1-1 \rangle = \langle -2, 4, 0 \rangle$

3) NORMAL VECTOR

$$\vec{N} = \underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ -4 & 3 & -1 \\ -2 & 4 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} i - \begin{vmatrix} -4 & -1 \\ -2 & 0 \end{vmatrix} j + \begin{vmatrix} -4 & 3 \\ -2 & 4 \end{vmatrix} k$$

$$= \langle 0+4, -(0-2), -16+6 \rangle$$

$$= \langle 4, 2, -10 \rangle$$

Equation: $X = -\frac{1}{2}Y + \frac{5}{2}Z - \frac{11}{2}$

4) EQUATION

POINT : $(0,-6,1)$, NORMAL VECTOR : $\langle 4, 2, -10 \rangle$

$4(x-0) + 2(y+6) - 10(z-1) = 0$

$4x + 2y + 12 - 10z + 10 = 0$

$4x = -2y + 10z - 10 - 12 \Rightarrow 4x = -2y + 10z - 22$

$x = -\frac{2}{4}y + \frac{10}{4}z - \frac{22}{4}$

5. *(10 = 6 + 4 points)* A particle has velocity vector $\mathbf{v}(t) = \langle 4t, \sin(t), 2e^{-2t}\rangle$ and initial position $\mathbf{r}(0) = \langle 2, 3, 0\rangle$.

(a) Find the position vector $\mathbf{r}(t)$

(b) Suppose further that the particle has mass $m = 3$. Find the force $\mathbf{F}(0)$ acting on the particle when $t = 0$

(a) $r(t) = \langle \int 4t\,dt, \int \sin(t)\,dt, \int 2e^{-2t}\,dt \rangle$

$= \langle 2t^2 + A, -\cos(t) + B, -e^{-2t} + C \rangle$

$r(0) = \langle 2(0)^2 + A, -\cos(0) + B, -e^{-2(0)} + C \rangle$

$= \langle A, -1 + B, -1 + C \rangle = \langle 2, 3, 0 \rangle$

$\Rightarrow \begin{cases} A = 2 \\ -1 + B = 3 \\ -1 + C = 0 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = 4 \\ C = 1 \end{cases} \Rightarrow r(t) = \langle 2t^2 + 2, -\cos(t) + 4, 2e^{-2t} + 1 \rangle$

(b) BY NEWTON'S SECOND LAW, $F(t) = m\,a(t)$

$a(t) = v'(t) = \langle 4, \cos(t), -4e^{-2t} \rangle$

$$\boxed{\mathbf{r}(t) = \langle 2t^2 + 2, -\cos(t) + 4, -e^{-2t} + 1 \rangle}$$

$$\boxed{\mathbf{F}(0) = \langle 12, 3, -12 \rangle}$$

HENCE $F(0) = 3\,a(0)$

$= 3 \langle 4, \cos(0), -4e^{-2(0)} \rangle$

$= 3 \langle 4, 1, -4 \rangle$

$= \langle 12, 3, -12 \rangle$