1.

Let P_n be the proposition: $s_n \leq 2$

Base Case: $s_1 = 2 \leq 2$ ✓

Inductive Step: Suppose P_n is true that is $s_n \leq 2$. Show P_{n+1} is true, that is $s_{n+1} \leq 2$.

However:

$$s_{n+1} = \sqrt{2 + s_n} \leq \sqrt{2 + 2} = \sqrt{4} = 2\checkmark$$

Where in the second step we used our inductive hypothesis.

Hence P_{n+1} is true and so P_n is true for all n, that is $s_n < 2$ for all n. $\square$

2.

Suppose the statement is false, that is there is $x \in \mathbb{R}$ such that for all $k \in \mathbb{Z}$, $k \leq x$.

Let $S = \{k \mid k \in \mathbb{Z}\}$ $(= \mathbb{Z})$

By assumption, S is bounded above by x and therefore, S has a least upper bound $M = Sup(S)$.

Consider $M - 1$. Since $M - 1 < M = \sup(S)$, hence there is $k \in S$ such that $k > M - 1 \Rightarrow M < k + 1$.

But then since $k + 1 \in S$, we found an element in S that is $> M$, which contradicts the fact that M is an upper bound of $S \Rightarrow\Leftarrow$ $\square$

3.

STEP 1: Scratchwork:

$$|f(s_n) - f(s)| \leq C\,|s_n - s| < \epsilon \Rightarrow |s_n - s| < \frac{\epsilon}{C}$$

STEP 2: Actual Proof:

Let $\epsilon > 0$ be given.

Then, since $s_n \to s$, there is N such that if $n > N$, then $|s_n - s| < \frac{\epsilon}{C}$.

But then, with that same N, if $n > N$, then

$$|f(s_n) - f(s)| \leq C\,|s_n - s| < C\left(\frac{\epsilon}{C}\right) = \epsilon \checkmark$$

Therefore $f(s_n) \to f(s)$ $\square$

4.

Let $S = \{s_n \mid n \in \mathbb{N}\}$

First of all, since (s_n) is bounded above by M, $s_n \leq M$ for all n, and therefore S is bounded above by M.

To show $\sup(S) = M$, suppose $M_1 < M$ and find $s_n \in S$ such that $s_n > M_1$

Let $\epsilon > 0$ TBA, then since $s_n \to M$, there is N such that if $n > N$, then

$$|s_n - M| < \epsilon \Rightarrow s_n - M > -\epsilon \Rightarrow s_n > M - \epsilon$$

Choose $\epsilon > 0$ such that $M - \epsilon \geq M_1$, that is $\epsilon \leq M - M_1$.

Then for all $n > N$, we have:

$$s_n > M - \epsilon \geq M - (M - M_1) = M_1 \Rightarrow s_n > M_1$$

Hence there is at least one s_n such that $s_n > M_1$ (in fact infinitely many of them), and therefore $\sup(S) = M$ $\square$