MATH 409 – MIDTERM 1 – STUDY GUIDE

The midterm will take place on **Thursday, September 30, 2021** from 12:45 pm to 2 pm in our usual lecture room. It is a closed book and closed notes exam, and counts for 20% of your grade. It will be an in-person exam and **NO** books, notes, calculators, cheat sheets will be allowed. Please bring your student ID card (or other government ID), for verification purposes. There will be 5 questions in total (with multiple parts), so expect to spend roughly 15 mins per question.

This is the study guide for the exam, and is just meant to be a *guide* to help you study, just so we're on the same place in terms of expectations. For a more thorough study experience, look at all the lecture notes, homework, and practice exams.

This exam covers everything up to and **including** section 10, with the *exception* of section 6 (Cuts), which won't be on the exam. The most important sections to focus on are sections 4, 8, 9, and 10 although I could ask you about the other sections as well. In section 10, I will not ask for Cauchy sequences or Completeness, but it might be part of the next midterm.

Expect the questions to be a mix of: definitions (see list below), one of the "Proofs you should know" below, providing counterexamples, proving a result similar to the homework, as well as new problems.

Date: Thursday, September 30, 2021.

Proofs you should know

Know how to prove the following theorems. I could theory ask you to reprove any of the below (or variations thereof):

(1) $\inf(S) = -\sup(-S)$

(2) Archimedean Property

(3) Limits are Unique

(4) Any of the 10 Examples of Limits in Lectures 5-7 (see sections 8 and 9 below)

(5) The Squeeze Theorem (Problem 8.5a)

(6) (s_n) converges $\Rightarrow$ (s_n) is bounded

(7) Limit laws such as $s_n + t_n \to s + t$ or $s_n t_n \to st$ or $\frac{t_n}{s_n} \to \frac{t}{s}$

(8) Infinite Limit Laws such as if $s_n \to \infty$ and $t_n \to \infty$, then $s_n t_n \to \infty$; or if $s_n \to \infty$ and $t_n \geq m$ for some m, then $s_n + t_n \to \infty$ (Problem 9.11c)

(9) The Duality Formula (Lecture 7)

(10) Monotone Sequence Theorem

(11) $\liminf s_n = -\left(\limsup -s_n\right)$

Definitions you should know

Know how to define the following concepts. I could in theory ask you to define them on the exam.

(1) Triangle Inequality and Reverse Triangle Inequality

(2) $\max(S)$ and $\min(S)$

(3) M is an upper bound for S, m is a lower bound for S

(4) $\sup(S) = M$, $\inf(S) = m$

(5) Least Upper Bound Property

(6) $\sup(S) = \infty$, $\inf(S) = -\infty$

(7) Archimedean Property

(8) $\mathbb{Q}$ is dense in $\mathbb{R}$.

(9) (s_n) converges to s

(10) The Squeeze Theorem

(11) $s_n \to \infty$, $s_n \to -\infty$

(12) Increasing/Decreasing

(13) $\limsup s_n$ and $\liminf s_n$

(14) $\limsup$ squeeze theorem

1. Section 1: The set $\mathbb{N}$ of natural numbers

- You **DON'T** need to know Peano's Axioms for $\mathbb{N}$, but know how to use them. Same goes for the Induction Axiom
- Although I won't specifically ask about it, understand the Mini Analysis Proof given in Lecture 1, it gives a good taste of what analysis proofs look like
- **Know how to prove a statement by induction**. I could very well ask you a non-analysis related question that uses induction
- The examples in Lecture 1, as well as Problems 1.9, AP1, and AP2 in HW1 are all excellent practice problems with induction. If you're feeling adventurous, you could even try problem 1.12
- Also check out AP3 in HW1, it's a good "Find a counterexample" exercise

2. Section 2: The set $\mathbb{Q}$ of rational numbers

- Know how to show that $\sqrt{2}$ is irrational. It's a classical proof that is hopefully familiar to you from previous courses.
- Know what it means for a number to be algebraic
- Show that a number like $\sqrt{\frac{\sqrt{3}-1}{2}}$ is algebraic and irrational
- You can ignore everything that has to do with the rational roots theorem
- Ignore AP1 on HW 2, I'm not going to ask anything about equivalence relations

3. Section 3: The set $\mathbb{R}$ of real numbers

- Do **NOT** memorize the axioms of a field, but know how to use them. In particular, you should be able to prove all the properties listed in the first Theorem in Lecture 2
- Similarly, do **NOT** memorize the axioms of an ordered field, but know how to use them and how to prove all the properties in the second Theorem in Lecture 2
- Know the definition of $|x|$
- **Know the Triangle Inequality and know how to use it! It's literally one of the most important tools in this course**
- Understand the proof of the corollary to the triangle inequality (with $\mathrm{dist}(a, b)$), it illustrates an important technique that's used over and over again
- Know the statement and of the reverse triangle inequality (Problem 3.5)
- Know how to do AP2ab in HW 2 but ignore AP2c
- Also check out 3.8 in HW 2

4. Section 4: The Completeness Axiom

- This is one of the most important sections for the midterm (along with sections 8, 9, and 10)

- Define the concept of max and min and show that S has a max or doesn't have a max or a min. The examples in the lecture are excellent practice examples
- Know how to show (or not) that S is bounded above (or below)
- Define $\sup(S)$ and $\inf(S)$ and show that $\sup(S) = M$ (or $\inf(S) = m$). The examples in Lecture 3, as well as AP3 in HW 2 are excellent practice with that.
- Know the statement of the least upper bound property
- Know how to show that $\inf(S) = -\sup(-S)$, and deduce the greatest lower bound property from that
- Know the statement *and* the proof of the Archimedean property and know how to use it
- Know the statement of $\mathbb{Q}$ dense in $\mathbb{R}$. No need to memorize the proof, but carefully note how the Archimedean property is used there
- Also check out Problems 4.7, 4.8, 4.14 (important), 4.16 and AP4 in HW 2. If you want more practice, also check out 4.10 and 4.15

5. Section 5: The Symbols ∞ and $-\infty$

This section is super short. Just know that $\sup(S) = \infty$ means S is not bounded above and $\sup(S) = -\infty$ means S is not bounded below. If you want more practice, check out AP3(c) from HW2 or check out 5.2 (prove those statements)

6. Section 6: A Development of $\mathbb{R}$

IGNORE this section, it will **NOT** be on the exam. One might even say I *cut* it out from the exam material ☺

7. Section 7: Limits of Sequences

The only important thing in that section is the definition of a limit (Definition 7.1 or Lecture 4) and the fact that limits are unique (Lecture 6). You don't need to know the definition of a sequence. But check out Problem 7.4, it's neat!

8. Section 8: A Discussion about Proofs

- This is the second important section to know for the midterm.
- Know how to do **ALL** the examples in this section and the lectures, they are all fair game and good practice with the definition of a limit. The examples in lecture include
  - Example 1: The Basics, $s_n = 3 - \frac{1}{n^2}$
  - Example 2: Simple Fraction, $s_n = \frac{2n+4}{4n+5}$
  - Example 3: Complex Fraction, $s_n = \frac{2n^3+3n}{n^3-2}$
  - Example 4: The Limit Does Not Exist, $s_n = (-1)^n$
  - Example 5: Square roots, $s_n \to s \Rightarrow \sqrt{s_n} \to \sqrt{s}$
  - Example 6: $s_n \to s \Rightarrow |s_n| \to |s|$ (see AP3 in HW 3)
- **Note:** It's important to write down **BOTH** the scratch work and the actual proof, otherwise you WILL lose points!

- Also know the statement about sequences that are bounded away from 0 and carefully note how it's used to prove quotient of limits.
- Know the statement and the proof of the Squeeze Theorem (See problem 8.5)
- Of course, problems 8.1, 8.2, 8.3, 8.4, 8.7, 8.8, 8.9, and 8.10 are excellent practice problems

9. Section 9: Limit Laws for Sequences

- Prove limit laws such as (most of those are in Lecture 6)
  - ▶ If $s_n \to s$ and $t_n \to t$, then $s_n + t_n \to s + t$
  - ▶ If $s_n \to s$ and $t_n \to t$ then $s_n t_n \to st$
  - ▶ If $s_n \neq 0$ and $s_n \to s \neq 0$ and $t_n \to t$, then $\frac{t_n}{s_n} \to \frac{t}{s}$; you'd of course have to show the intermediate step of Example 7 with $\frac{1}{s_n}$
- Know how to show that if (s_n) converges, then (s_n) is bounded
- Know how to show
  - ▶ Example 7: If $s_n \neq 0$ and $s_n \to s \neq 0$, then $\frac{1}{s_n} \to \frac{1}{s}$
  - ▶ Example 8: If $|a| < 1$, then $\lim_{n\to\infty} a^n = 0$
  - ▶ Example 9: $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ and its corollary $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ if $a > 0$
  - ▶ Example 10: Infinite limits such as $\lim_{n\to\infty} \sqrt{n-2}+3 = \infty$

- **Note:** Even though I haven't explicitly done it, know how to show that $\lim_{n\to\infty} \frac{1}{n^p} = 0$ if $p > 0$ (This is just 8.1(d) but with p instead of 3)

- Know the binomial theorem. You won't need the full version, just the version with:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + b^n$$

- Know how to define $\lim_{n\to\infty} s_n = \infty$ and $\lim_{n\to\infty} s_n = -\infty$ and know how to show that a sequence goes to ∞ (like Example 10 above)

- Prove some limit laws for infinite limits, such as if $s_n \to \infty$ and $t_n \to \infty$, then $s_n t_n \to \infty$ if $s_n \to \infty$ and $t_n \geq m$ for some m, then $s_n + t_n \to \infty$ (Problem 9.11(c))

- Know the statement and the proof of the Duality Formula (Lecture 7), and use it to show for example that $\lim_{n\to\infty} 2^n = \infty$

- Problems 9.6(a)(b), 9.9, 9.10, 9.11, 9.17, the AP in HW 3 and AP 1, AP 2 in HW 4 are great practice problems

- You can ignore problem 9.12

Section 10: Monotone Sequences and Cauchy Sequences

- Define: increasing/decreasing

- Prove the Monotone Sequence Theorem (Theorem 10.2), as well as its corollaries: "If (s_n) is decreasing and bounded below, then it converges," and "If (s_n) is increasing, then it either converges or goes to ∞"

- Ignore the discussion about Decimal Expansions
- Define: $\limsup_{n\to\infty} s_n$ and explain why $\limsup$ exists (basically because $v_N = \sup\{s_n \mid n > N\}$ is decreasing and bounded below); same with $\liminf$
- Find, with proof, $\limsup_{n\to\infty}(-1)^n$ and $\liminf_{n\to\infty}(-1)^n$
- Prove $\liminf s_n = -(\limsup -s_n)$. This is a very important identity that allows us to go from $\limsup$ to $\liminf$
- Two important facts about $\limsup$ (no need to know the proof)
  (1) If $s_n \to s$, then $\liminf s_n = \limsup s_n = s$
  (2) Limsup squeeze theorem: If $\liminf s_n = \limsup s_n = s$, then $s_n \to s$
- You can ignore everything about Cauchy sequences for now
- I recommend checking out AP3, AP4, AP5, and AP6 on HW 4