MATH 409 – MIDTERM 2 – SOLUTIONS

1. Option 1: The Limsup Product Rule

There is a subsequence (t_{n_k}) of t_n that converges to $t =: \limsup_{n\to\infty} t_n$

But then $(s_{n_k} t_{n_k})$ is a subsequence of $(s_n t_n)$ that converges to st.

Since $\limsup_{n\to\infty} s_n t_n$ is the *largest* limit point of $s_n t_n$, we have

$$\limsup_{n\to\infty} s_n t_n \geq st = \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right)$$

On the other hand, $\frac{1}{s_n} \to \frac{1}{s} > 0$, so by the result above, we have

$$\limsup_{n\to\infty} t_n = \limsup_{n\to\infty} \left(\frac{1}{s_n}\right) s_n t_n \geq \left(\limsup_{n\to\infty} \frac{1}{s_n}\right)\left(\limsup_{n\to\infty} s_n t_n\right) = \frac{1}{s} \limsup_{n\to\infty} s_n t_n$$

Hence $\limsup_{n\to\infty} s_n t_n \leq s \limsup_{n\to\infty} t_n = \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right)$

Combining the two results, we get

$$\limsup_{n\to\infty} s_n t_n = \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right)$$

Date: Thursday, November 4, 2021.

Option 2: Continuous functions are bounded

Suppose not, then for all $n \in \mathbb{N}$ there is $x_n \in [a,b]$ such that $f(x_n) \geq n$.

But since x_n is bounded, by the Bolzano Weierstrass Theorem, there is a subsequence (x_{n_k}) that converges to some $x_0 \in [a,b]$

Since $x_{n_k} \to x_0$ and f is continuous, then $f(x_{n_k}) \to f(x_0)$ and so $|f(x_{n_k})| \to |f(x_0)|$

On the other hand $|f(x_n)| \geq n$ for all n, and so $|f(x_{n_k})| \geq n_k$ for all k, and letting k go to infinity we get $|f(x_{n_k})| \to \infty$.

But then, comparing limits, we get $|f(x_0)| = \infty$, which is a contradiction $\Rightarrow\Leftarrow$

2. (a) For all $\epsilon > 0$ there is $\delta > 0$ such that for all x, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$

(b) **STEP 1:** Scratchwork

$$|f(x) - f(x_0)| = \left|3x^2 - 5 - \left(3(x_0)^2 - 5\right)\right| = \left|3x^2 - 3(x_0)^2\right| = 3|x - x_0||x + x_0|$$

Now if $|x - x_0| < 1$, then

$$|x + x_0| = |x - x_0 + x_0 + x_0| = |x - x_0 + 2x_0| \leq |x - x_0| + 2|x_0| = 1 + 2|x_0|$$

And therefore

$$|f(x) - f(x_0)| = 3|x - x_0||x + x_0| \leq 3|x - x_0|(1 + 2|x_0|) = 3(1 + 2|x_0|)|x - x_0| < \epsilon$$

Which gives $|x - x_0| = \frac{\epsilon}{3(1+2|x_0|)}$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{3(1+2|x_0|)}$, then if $|x - x_0| < \delta$, then $|x + x_0| \leq 1 + 2|x_0|$, so

$$\begin{aligned} |f(x) - f(x_0)| = 3|x - x_0||x + x_0| &\leq 3(1 + 2|x_0|)|x - x_0| \\ &< 3(1 + 2|x_0|)\left(\frac{\epsilon}{3(1 + 2|x_0|)}\right) = \epsilon \end{aligned}$$

Hence f is continuous at x_0

3. (a) $\sum a_n$ converges if and only if for all $\epsilon > 0$ there is N such that for all m, n, if $n \geq m > N$, then $|\sum_{k=m}^{n} a_k| < \epsilon$

(b) Let $\epsilon > 0$ be given.

Since $\sum a_n$ converges, by the Divergence Test, $a_n \to 0$, so there is N_1 such that if $n > N_1$, then $|a_n| < 1$ that is, $a_n < 1$

Since $\sum a_n$ converges, by the Cauchy criterion, there is N_2 such that if $n \geq m > N_2$, then $\sum_{k=m}^{n} a_k < \epsilon$ (no absolute value since the terms are positive)

Let $N = \max\{N_1, N_2\}$, then if $n \geq m > N$, then

$$\left| \sum_{k=m}^{n} (a_k)^2 \right| = \sum_{k=m}^{n} (a_k)^2 \leq \sum_{k=m}^{n} a_k < \epsilon$$

Where in the second step we used $(a_k)^2 < a_k$ since $a_k < 1$

Hence by the Cauchy Criterion, $\sum (a_n)^2$ converges as well

(c) Let $a_n = \frac{(-1)^n}{\sqrt{n}}$, then $\sum a_n$ converges by the alternating series test, but $\sum (a_n)^2 = \sum \frac{1}{n}$ diverges by the 1−series

(d) Let $a_n = \frac{1}{n}$, then $\sum a_n = \sum \frac{1}{n}$ diverges by the 1−series, but $\sum (a_n)^2 = \sum \frac{1}{n^2}$ converges by the 2−series

4. We will construct a decreasing sequence (s_n) with the property that for all n,

$$\inf(S) < s_n < \inf(S) + \frac{1}{n}$$

Then by the squeeze theorem $s_n \to \inf(S)$ and we would be done

Base Case: Consider $\inf(S) + 1 > \inf(S)$, so by definition of inf, there is some $s_1 \in S$ with $s_1 < \inf(S) + 1$. On the other hand, since $\inf(S)$ is a lower bound for S, we get $s_1 \geq \inf(S)$, and in fact $s_1 > \inf(S)$ because otherwise $\inf(S) = s_1 \in S$. Therefore $\inf(S) < s_1 < \inf(S) + 1$ ✓

Inductive Step: Suppose we found $s_1 > s_2 > \cdots > s_n$ with $\inf(S) < s_k < \inf(S) + \frac{1}{k}$ for all $k = 1, \cdots, n$

Find $s_{n+1} < s_n$ such that $\inf(S) < s_{n+1} < \inf(S) + \frac{1}{n+1}$

Let $m = \min\left\{\inf(S) + \frac{1}{n+1}, s_n\right\} > \inf(S)$

Then by definition of $\inf(S)$ there is $s_{n+1} \in S$ such that $s_{n+1} < m$, but this implies $s_{n+1} < s_n$ (so (s_n) is decreasing) and $s_{n+1} < \inf(S) + \frac{1}{n+1}$. Moreover, since $\inf(S)$ is a lower bound for S, we have $s_{n+1} \geq \inf(S)$, and in fact $s_{n+1} > \inf(S)$ because otherwise $\inf(S) = s_{n+1} \in S$. Therefore $\inf(S) < s_{n+1} < \inf(S) + \frac{1}{n+1}$ ✓