SOLUTIONS

MATH 251 – MIDTERM 2

Name	
Student ID	
Section	501
Signature	

Instructions: Welcome to your Midterm! You have 50 minutes to take this exam, for a total of 50 points. No books, notes, calculators, or cellphones are allowed. Remember that you are not only graded on your final answer, but also on your work. **Please put your answers in the boxes provided.** If you need to continue your work on a scratch paper, please check the box "Work on Scratch Paper," or else your work will be discarded.

Academic Honesty Statement: With the signature above, I certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating (no matter how small) results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to the Aggie Honor Code.

Date: Friday, October 15, 2021.

1. *(10 points)* Find the equation of the tangent plane **and** the **vector** equation of the normal line to the following surface at $(1,1,1)$

$$x\left(z^3\right) - x\left(y^2\right) - yz + z^2 = 0$$

$F(x,y,z) = xz^3 - xy^2 - yz + z^2$

$\nabla F(x,y,z) = \langle F_x, F_y, F_z \rangle = \langle z^3 - y^2, -2xy - z, 3xz^2 - y + 2z \rangle$

$\nabla F(1,1,1) = \langle 1-1, -2-1, 3-1+2 \rangle = \langle 0, -3, 4 \rangle$

POINT $(1,1,1)$

TANGENT PLANE $0(x-1) - 3(y-1) + 4(z-1) = 0$

NORMAL LINE $x(t) = 1 + 0t$

$y(t) = 1 - 3t$

$z(t) = 1 + 4t$

Tangent Plane	$-3(y-1) + 4(z-1) = 0$
Normal line	$r(t) = \langle 1, 1-3t, 1+4t \rangle$

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2. *(10 points)* The volume of a Peyamagon is given by

$$V = \frac{1}{4}\pi\left(r^2\right)h\left(w^4\right)$$

Suppose you're trying to measure the volume of a Peyamagon with $r = 2$, $h = 4$ and width $w = 1$, but your errors in measurement are $dr = 0.1$, $dh = -0.2$ and $dw =$ [redacted]. -0.1

Approximate the error in finding the volume; express your answer in terms of π

$$\Delta V \approx dV = V_r\,dr + V_h\,dh + V_w\,dw$$

$$= \left(\frac{\pi}{4}(2r)\,h\,w^4\right)(dr) + \left(\frac{\pi}{4}r^2 w^4\right)dh + \left(\frac{\pi}{\cancel{4}}r^2 h\,\cancel{4}w^3\right)dw$$

$$= \frac{\pi}{\cancel{2}}(\cancel{2})(4)(1^4)(0.1) + \frac{\pi}{\cancel{4}}(\cancel{2^2})(1^4)(-0.2) + \pi(2^2)(4)(1^3)(-0.1)$$

$$= 0.4\pi + (-0.2)\pi + (-1.6)\pi$$

$$= (-1.4)\pi$$

Error in Volume	$\Delta V \approx (-1.4)\pi$

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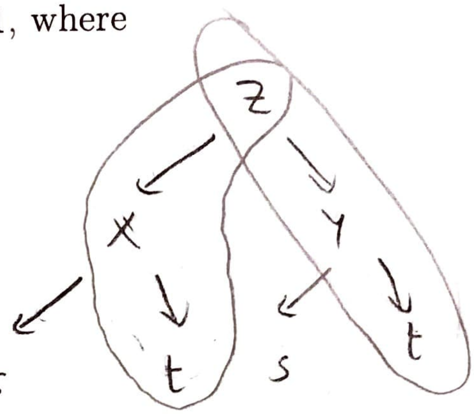
3. *(10 points)* Find $\frac{\partial z}{\partial t}$ at $s = 2$ and $t = 1$, where

$$\begin{cases} z = x^3 + xy + y^4 \\ x = \dfrac{s}{t} \\ y = st \end{cases}$$



$$\frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right)$$

$$= (x^3 + xy + y^4)_x \left(\frac{s}{t}\right)_t + (x^3 + xy + y^4)_y \, (st)_t$$

$$= (3x^2 + y)\left(\frac{-s}{t^2}\right) + (x + 4y^3)(s)$$

BUT IF $s = 2$ AND $t = 1$ THEN $x = \frac{s}{t} = \frac{2}{1} = 2$ AND $y = st = (2)(1) = 2$

$$= (3(2)^2 + 2)\left(-\frac{2}{1^2}\right) + (2 + 4(2^3))(2)$$

$$= (14)(-2) + (34)(2) = -28 + 68 = 40$$

$\frac{\partial z}{\partial t} =$	40

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$(*)$ $(-1,-1)$ $D(-1,-1) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = (12)^2 - (-4)^2 = 144 - 16 = 128 > 0$

AND $f_{xx} = 12 > 0$ LOCAL MIN

$f(-1,-1) = 1 - 4 + 1 = -2$

4. *(10 points)* Find the local maximum/minimum/saddle point **values** of f and the point(s) at which they occur (if they exist).

$$f(x, y) = x^4 - 4xy + y^4$$

1) CRITICAL POINTS $\begin{cases} f_x = 4x^3 - 4y = 0 \Rightarrow x^3 = y \Rightarrow y = x^3 \\ f_y = -4x + 4y^3 \Rightarrow x = y^3 = (x^3)^3 = x^9 \end{cases}$

SO $x - x^9 = 0 \Rightarrow x(1 - x^8) = 0 \Rightarrow x = 0$ OR $x^8 = 1$
$\Rightarrow x = 0$ OR $x = \pm 1$

IF $x = 0$, THEN $y = x^3 = 0^3 = 0 \Rightarrow (0,0)$
IF $x = 1$ THEN $y = x^3 = 1^3 = 1 \Rightarrow (1,1)$
IF $x = -1$ THEN $y = x^3 = (-1)^3 = -1 \Rightarrow (-1,-1)$

2) SECOND DERIVATIVES $D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix}$

Local maximum	DNE	at	DNE
Local minimum	-2	at	$(1,1)$ AND $(-1,-1)$
Saddle Point	0	at	$(0,0)$

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$(0,0)$ $D(0,0) = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = 0^2 - (-4)^2 = -16 < 0$ SADDLE POINT

$f(0,0) = 0$

$(1,1)$ $D(1,1) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = (12)^2 - (-4)^2 = 144 - 16 = 128 > 0$

AND $f_{xx} = 12 > 0$ LOCAL MIN

$f(1,1) = 1 - 4 + 1 = -2$ $(*)$

AND IF $x=-1$ THEN $x^2+\frac{y^2}{2}=4 \Rightarrow 1+\frac{y^2}{2}=4 \Rightarrow \frac{y^2}{2}=3 \Rightarrow$

$\Rightarrow y^2=6 \Rightarrow y=\pm\sqrt{6}$

WHICH GIVES $(-1,-\sqrt{6})$ AND $(-1,\sqrt{6})$

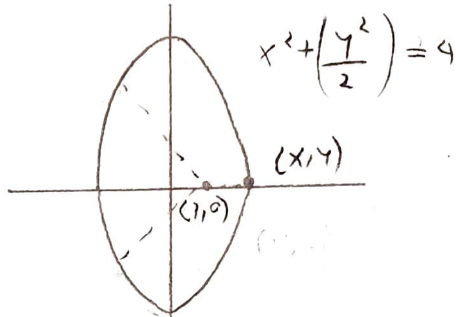
5) $f(2,0)=(2-1)^2+0^2=1$ $\quad f(-1,-\sqrt{6})=(-2)^2+(\sqrt{6})^2=4+6=10$

$f(-1,\sqrt{6})=(-2)^2+(\sqrt{6})^2=10$

$f(-2,0)=(-2-1)^2+0^2=9$

5. *(10 points)* Use **Lagrange multipliers** to find the point(s) on the ellipse $x^2+\left(\frac{y^2}{2}\right)=4$ that are closest and furthest from $(1,0)$

1) PICTURE (OPTIONAL)



2) $f(x,y)=(\text{DISTANCE})^2=(x-1)^2+(y-0)^2=(x-1)^2+y^2$

$g(x,y)=x^2+\frac{y^2}{2}-4$

3) LAGRANGE EQUATION $\quad \nabla f=\lambda \nabla g$

$$\begin{cases} f_x=\lambda g_x \\ f_y=\lambda g_y \end{cases} \Rightarrow \begin{cases} 2(x-1)=\lambda(2x) \\ 2y=\lambda\left(\frac{2y}{2}\right) \\ x^2+(y^2/2)=4 \end{cases} \Rightarrow \begin{cases} x-1=\lambda x \\ 2y=\lambda y \\ x^2+(y^2/2)=4 \end{cases}$$

Closest point(s)	$(2,0)$
Furthest point(s)	$(-1,-\sqrt{6})$ AND $(-1,\sqrt{6})$

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LOOK AT $2y=\lambda y$

$\Rightarrow 2y-\lambda y=0 \Rightarrow y(2-\lambda)=0 \Rightarrow y=0$ OR $\lambda=2$

4) CASE 1 $\quad y=0$ THEN $x^2+(y^2/2)=4 \Rightarrow x^2+0=4 \Rightarrow x=\pm 2$

WHICH GIVES $(2,0)$ AND $(-2,0)$

CASE 2 $\quad \lambda=2$ THEN $x-1=\lambda x \Rightarrow x-1=2x \Rightarrow x=-1$ (*)