MAT 267 – MIDTERM 2 – SOLUTIONS

1. Multiple Choice

(1) $\boxed{\textbf{C}}$ Since $\ln(x)$ is defined only for $x > 0$, the domain of $f(x, y) = \ln(x) + \ln(y)$ is the all the points (x, y) with $x > 0$ and $y > 0$, which is the **first quadrant** in the xy plane

(2) $\boxed{\textbf{B}}$ The level curve $z = 1$ is the curve $6x^2 + 3y^2 = 1$, which is an **ellipse**.

(3) $\boxed{\textbf{D}}$ The equation of the tangent plane to $f(x, y) = 3x^2 - \frac{1}{y}$ at the point $(2, -1)$ is

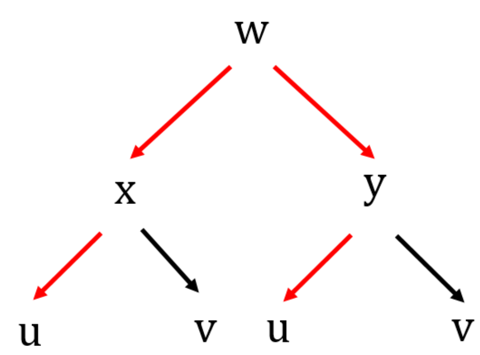
$$z - f(2, -1) = f_x(2, -1)(x - 2) + f_y(2, -1)(y + 1)$$

$$\begin{aligned} f(2, -1) &= 3(2)^2 - \left(\frac{1}{-1}\right) = 12 + 1 = 13 \\ f_x(x, y) &= 6x \Rightarrow f_x(2, -1) = 6(2) = 12 \\ f_y(x, y) &= \frac{1}{y^2} \Rightarrow f_y(2, -1) = \frac{1}{(-1)^2} = 1 \end{aligned}$$

Hence the equation becomes:

$$\begin{aligned} z - 13 =& 12(x-2) + 1(y+1) \\ z - 13 =& 12x - 24 + y + 1 \\ z =& 12x + y - 23 + 13 \\ z =& 12x + y - 10 \end{aligned}$$

(4) **C**



By the Chen Lu, we have:

$$\begin{aligned} \frac{\partial w}{\partial u} &= \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) \\ &= (2xy - 3y)_x (3u - 2v)_u + (2xy - 3y)_y \left(v^2 \sin(u)\right)_u \\ &= (2y)(3) + (2x - 3)\left(v^2 \cos(u)\right) \end{aligned}$$

(5) **A** Let $F(x, y, z) = x^2yz^4 - 2z - 1$. Then a normal vector is simply $\nabla F(1, 3, 1)$

$$\nabla F = \langle F_x, F_y, F_z \rangle = \left\langle 2xyz^4, x^2z^4, x^2y(4z^3) - 2 \right\rangle$$

Therefore at the point $(1,3,1)$ we have:

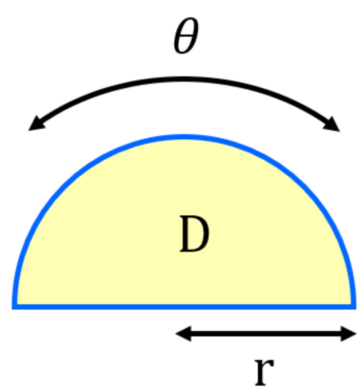
$$\nabla F = \left\langle 2(1)(3)\left(1^4\right), \left(1^2\right)\left(1^4\right), \left(1^2\right)(3)4(1^3) - 2\right\rangle = \langle 6, 1, 10\rangle$$

(6) $\boxed{\mathbf{B}}$ The region of integration becomes:

$$\begin{aligned} 0 \le y \le \sqrt{9-x^2} \\ -3 \le x \le 3 \end{aligned}$$

Since $y = \sqrt{9-x^2} \Rightarrow y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9$ (circle).

The region of integration is the upper semicircle of in the plane centered at $(0,0)$ and radius 3:



In terms of polar coordinates D is written as

$$\begin{aligned} 0 \le & r \le 3 \\ 0 \le & \theta \le \pi \end{aligned}$$

Finally $\cos\left(x^2+y^2\right) = \cos\left(r^2\right)$ and so the integral becomes:

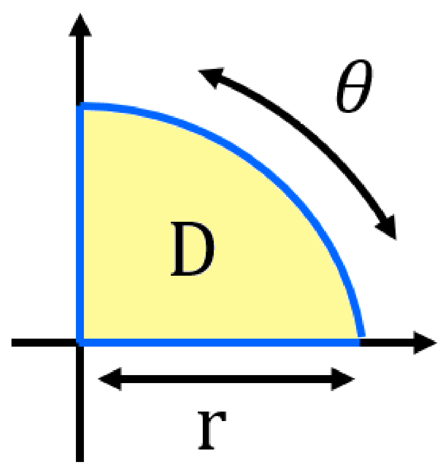
$$\int_0^{\pi} \int_0^3 \cos\left(r^2\right) r \, dr d\theta$$

(7) **D**

Function:

$$\left(x^2 + y^2\right)^{\frac{5}{2}} = \left(r^2\right)^{\frac{5}{2}} = r^5$$

Picture:



Inequalities:

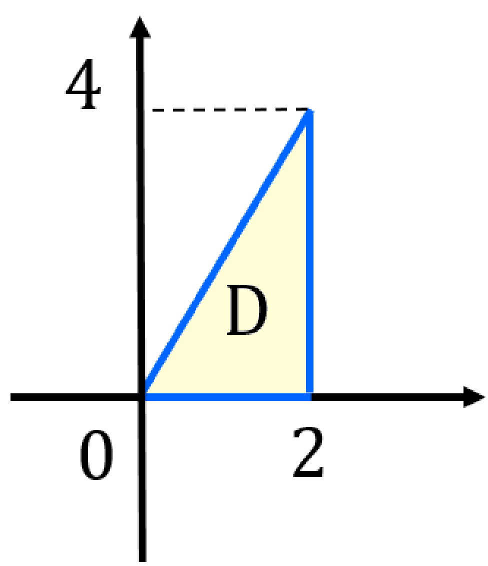
$$0 \leq r \leq 1$$
$$0 \leq \theta \leq \frac{\pi}{2}$$

Integrate:

$$\int_0^{\frac{\pi}{2}} \int_0^1 r^5 r \, dr d\theta$$
$$= \frac{\pi}{2} \int_0^1 r^6 dr$$
$$= \frac{\pi}{2} \left[\frac{r^7}{7} \right]_0^1$$
$$= \frac{\pi}{2} \left(\frac{1}{7} \right)$$
$$= \frac{\pi}{14}$$

(8) C

Draw D:



$$\int\int_D 1 dxdy = \text{ Area } (D) = \frac{1}{2} \times 2 \times 4 = 4$$

2. Free Response

1(a)

$$\begin{aligned}\nabla T &= \langle T_x, T_y, T_z\rangle = \left\langle 2xy-3z, x^2+2yz, y^2-3x\right\rangle \\ \nabla T(1,2,-1) &= \left\langle 2(1)(2)-3(-1), (1)^2+2(2)(-1), (2)^2-3(1)\right\rangle \\ &= \langle 4+3, 1-4, 4-3\rangle \\ &= \langle 7, -3, 1\rangle\end{aligned}$$

1(b)

$$\|\nabla T(1,2,-1)\| = \|\langle 7,-3,1\rangle\| = \sqrt{7^2+(-3)^1+1^2} = \sqrt{49+9+1} = \sqrt{59}$$

1(c) A **unit** vector in the direction of $\mathbf{v} = \langle 1,1,1\rangle$ is

$$\mathbf{v}' = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$$

Hence the directional derivative is:

$$\begin{aligned}\nabla T(1,2,-1)\cdot\mathbf{v}' &= \langle 7,-3,1\rangle \cdot \frac{1}{\sqrt{3}}\langle 1,1,1\rangle \\ &= \frac{1}{\sqrt{3}}\left((7)(1)+(-3)(1)+(1)(1)\right) \\ &= \frac{5}{\sqrt{3}}\end{aligned}$$

2(a)

$$\begin{aligned}f_x &= 3x^2-6y = 0 \\ f_y &= 2y-6x = 0 \Rightarrow 2y = 6x \Rightarrow y = 3x\end{aligned}$$

Plugging the second equation into the first, we get:

$$\begin{aligned} 3x^2 - 6y &= 0 \\ 3x^2 - 6(3x) &= 0 \\ x^2 - 6x &= 0 \\ x(x-6) &= 0 \end{aligned}$$

Which gives $x = 0$ or $x = 6$

Case 1: $x = 0$, then $y = 3x = 3(0) = 0$, which gives $(0, 0)$

Case 2: $x = 6$, then $y = 3x = 3(6) = 18$, which gives $(6, 18)$

Therefore there are two critical points: $(0, 0)$ and $(6, 18)$

2(b)

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -6 \\ -6 & 2 \end{vmatrix}$$

Case 1: For $(0, 0)$, we get:

$$D(0,0) = \begin{vmatrix} 0 & -6 \\ -6 & 2 \end{vmatrix} = (0)(2) - (-6)(-6) = -36 < 0$$

Hence $(0, 0)$ is a **saddle point**

Case 2: For $(6, 18)$, we get

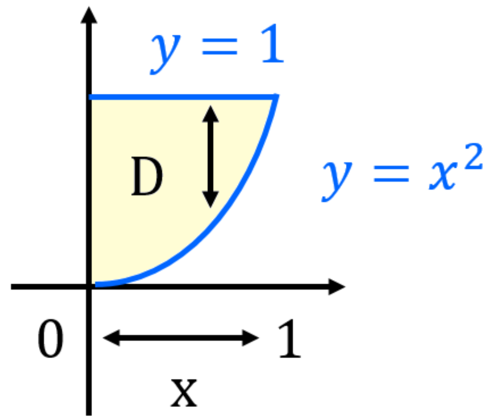
$$D(6,18) = \begin{vmatrix} 6(6) & -6 \\ -6 & 2 \end{vmatrix} = \begin{vmatrix} 36 & -6 \\ -6 & 2 \end{vmatrix} = (36)(2) - (-6)(-6) = 72 - 36 = 36 > 0$$

Moreover $f_{xx}(6, 18) = 36 > 0$, so f has a local **min** at (6,18).

3(a) **Inequalities:**

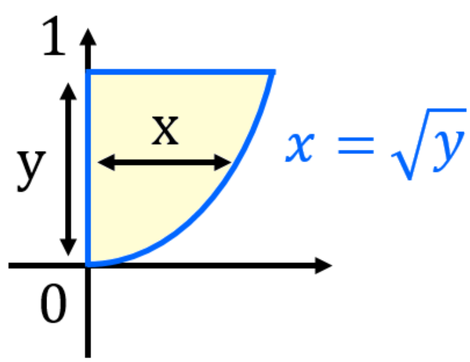
$$x^2 \leq y \leq 1$$
$$0 \leq x \leq 1$$

Picture:



3(b) We need to rewrite this as a horizontal region. For this, notice that

$$y = x^2 \Rightarrow x^2 = y \Rightarrow x = \sqrt{y}$$



$$\begin{aligned} \text{Left } \leq & x \leq \text{ Right} \\ 0 \leq & x \leq \sqrt{y} \end{aligned}$$

Therefore our inequalities become:

$$\begin{aligned} 0 \leq & x \leq \sqrt{y} \\ 0 \leq & y \leq 1 \end{aligned}$$

And our integral becomes:

$$\int_0^1 \int_0^{\sqrt{y}} \frac{xy}{1+y^3} dxdy$$

3(c)

$$\begin{aligned}
&\int_0^1 \int_0^{\sqrt{y}} x \left(\frac{y}{1+y^3} \right) dxdy \\
=&\int_0^1 \left[\left(\frac{x^2}{2} \right) \left(\frac{y}{1+y^3} \right) \right]_{x=0}^{x=y} dy \\
=&\int_0^1 \left(\frac{\left(\sqrt{y}\right)^2}{2} \right) \left(\frac{y}{1+y^3} \right) dy \\
=&\frac{1}{2} \int_0^1 \frac{y^2}{1+y^3} dy \\
=&\frac{1}{2} \left[\left(\frac{1}{3} \right) \ln \left| y^3+1 \right| \right]_0^1 \\
=&\frac{1}{2} \left(\frac{1}{3} \right) \left(\ln(1^3+1) - \ln(0^3+1) \right) \\
=&\frac{1}{6} \left(\ln(2) - \ln(1) \right) \\
=&\frac{\ln(2)}{6}
\end{aligned}$$