1.

STEP 1: By definition of $\sup(S) = \infty$ with $M = 1$, there is some $s_1 \in S$ with $s_1 > 1$ ✓

STEP 2: Suppose we defined $s_1, \dots s_n$ such that $s_1 < s_2 < \cdots < s_n$ and with $s_k > k$ for all $k = 1, \dots, n$.

Let $M = \max\{s_n, n+1\}$

Then, by definition of $\sup(S) = \infty$, there is some $s_{n+1} \in S$ with $s_{n+1} > M = \max\{s_n, n+1\}$. Therefore, we have both $s_{n+1} > s_n$ and $s_{n+1} > n+1$ ✓

STEP 3: Therefore, by the inductive process, we have constructed an increasing sequence (s_n) with $s_n > n$ for all n $\square$

2.

STEP 1: Since $\alpha > 1$, there is $\epsilon > 0$ with $\alpha - \epsilon > 1$. Now by definition

$$\liminf_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{N\to\infty} \inf \left\{ |a_n|^{\frac{1}{n}} \,\middle|\, n > N \right\} = \alpha$$

STEP 2: By definition of a limit, there is N_1 such that if $N > N_1$, then

$$\left| \inf \left\{ |a_n|^{\frac{1}{n}} \middle| n > N \right\} - \alpha \right| < \epsilon$$

Since this is true for all $N > N_1$, this is true for *some* N ($> N_1$). Therefore, in particular,

$$\begin{aligned} \inf \left\{ |a_n|^{\frac{1}{n}} \,\middle|\, n > N \right\} - \alpha > -\epsilon &\Rightarrow \inf \left\{ |a_n|^{\frac{1}{n}} \,\middle|\, n > N \right\} > \alpha - \epsilon \\ &\Rightarrow |a_n|^{\frac{1}{n}} > \alpha - \epsilon \text{ (for all } n > N) \\ &\Rightarrow |a_n| > (\alpha - \epsilon)^n \\ &\Rightarrow a_n > (\alpha - \epsilon)^n \text{ (since } a_n \geq 0) \end{aligned}$$

STEP 3: However, since

$$\sum_{n=N+1}^{\infty} (\alpha - \epsilon)^n = \infty$$

(Geometric series with $r = \alpha - \epsilon > 1$), by comparison we get that $\sum a_n = \infty$ $\square$

3.

(a) $E^\circ = \emptyset$

Suppose $x \in E^\circ$, then for some $r > 0$, $(x - r, x + r) \subseteq E$. But this is a contradiction because $(x - r, x + r)$ has some irrational numbers in it, whereas every element in E is rational $\Rightarrow\Leftarrow$

(b) $\overline{E} = E \cup \{2\}$

First of all, every $x \in E$ is in $\overline{E}$ (by considering the constant sequence $s_n = x$). Moreover, the sequence $s_n = 2 - \frac{1}{n}$ is in E and converges to 2, therefore $2 \in \overline{E}$. Since those are all the possible limit points of E, we get that $\overline{E} = E \cup \{2\}$

(c) $\partial E = E \cup \{2\}$

Using (a) and (b), we get

$$\partial E = \overline{E} \backslash E^\circ = \overline{E} \backslash \emptyset = \overline{E} = E \cup \{2\}$$

4.

STEP 1: Suppose $\bigcap_{n=1}^{\infty} F_n = \emptyset$, and consider

$$\mathcal{U} = \{F_1^c, F_2^c, \dots\}$$

First of all, each F_n is closed, therefore each F_n^c is open.

Moreover $\mathcal{U}$ covers S, because otherwise there would be some $s \in S$ that is in none of the F_n^c, so s must be in all the F_n, and so $s \in \bigcap_{n=1}^{\infty} F_n = \emptyset \Rightarrow\Leftarrow$[1]

STEP 2: Since S is compact and $\mathcal{U}$ covers S, there is a finite sub-cover $\mathcal{V}$ of S, where

$$\mathcal{V} = \left\{F_{n_1}^c, \dots, F_{n_N}^c\right\}$$

Let $M = \max\{n_1, \dots, n_N\}$. Since the F_n are non-increasing, the F_n^c are non-decreasing (because $A \subseteq B \Rightarrow B^c \subseteq A^c$), and therefore, since $\mathcal{V}$ covers S, we have

$$S \subseteq \bigcup_{k=1}^{N} F_{n_k}^c = F_M^c \Rightarrow S \subseteq F_M^c \Rightarrow F_M \subseteq S^c = \emptyset$$

But this implies that $F_M = \emptyset \Rightarrow\Leftarrow$ $\square$

[1] Or you can use

$$\bigcup_{n=1}^{\infty} F_n^c = \left(\bigcap_{n=1}^{\infty} F_n\right)^c = (\emptyset)^c = S$$