MATH 409 – MIDTERM 2 – STUDY GUIDE

The midterm will take place on **Thursday, November 4, 2021** from 12:45 pm to 2 pm in our usual lecture room. It is a closed book and closed notes exam, and counts for 20% of your grade. It will be an in-person exam and **NO** books, notes, calculators, cheat sheets will be allowed. Please bring your student ID card (or other government ID), for verification purposes. This time there will be only 4 questions (with multiple parts).

This is the study guide for the exam, and is just meant to be a *guide* to help you study, just so we're on the same place in terms of expectations. For a more thorough study experience, look at all the lecture notes, homework, and practice exams.

This exam covers everything starting from section 10 (from Cauchy sequences on), up to and including section 19. In section 19. Any material covered on Tuesday, November 2, will **not** be on the midterm.

The format of the midterm is the same as last time. There will be a mix of definitions, counterexamples, computational problems, problems similar to the homework and one of the "Proofs you should know" below.

Date: Thursday, November 4, 2021.

Proofs you should know

Know how to prove the following theorems. I could theory ask you to reprove any of the below (or variations thereof):

(1) (s_n) converges $\Rightarrow$ (s_n) Cauchy

(2) Limsup product rule (ignore the cases $t = \pm\infty$)

(3) Divergence Test and Comparison Test

(4) Root Test (just the case $\alpha < 1$)

(5) Integral Test

(6) The sequence and $\epsilon - \delta$ definitions of continuity are equivalent

(7) f is continuous implies f is bounded

(8) Extreme Value Theorem

(9) Intermediate Value Theorem

Definitions you should know

Know how to define the following concepts. I could in theory ask you to define them on the exam.

(1) Cauchy Sequence

(2) Completeness

(3) Limit Point

(4) Bolzano-Weierstrass Theorem

(5) lim inf and lim sup (again ☺)

(6) Series, partial sum, $\sum a_n$ converges/diverges

(7) Cauchy Criterion

(8) Divergence Test

(9) Comparison Test

(10) Root Test

(11) Ratio Test

(12) Integral Test

(13) f continuous (both the $\epsilon - \delta$ and the sequence definition)

(14) Extreme Value Theorem

(15) Intermediate Value Theorem

(16) Uniformly Continuous

Section 10: Monotone Sequences and Cauchy Sequences

- Define: Cauchy sequence. It's a useful way of studying sequences, without knowing their limit beforehand.
- Prove: (s_n) is converges $\Rightarrow$ (s_n) Cauchy

- Know that Cauchy sequences are bounded. Although I won't ask about the proof, notice how it's similar to the fact that convergenct sequences are bounded.

- Know the fact that Cauchy sequences in $\mathbb{R}$ are convergent; you don't need to know the proof, but notice how the Limsup Squeeze Theorem is used in it.

- Define: completeness and convince yourself that $\mathbb{Q}$ is not complete

- Show $\mathbb{Z}$ is complete (see AP1 in the section 10 homework)

Section 11: Subsequences

- Understand the concept of a subsequence, but you don't need to know the definition with σ.

- Notice that if s_n converges to s, then every subsequence s_{n_k} converges to s. This just uses the fact that $n_k \geq k$ (an express train is faster than the original train)

- **Know how to do an inductive construction.** The following examples are good practice with this concept:

  (1) Show there is an increasing sequence of rational numbers that converges to a given real number (see Inductive Construction 1 in the Subsequence Lecture)

  (2) If (s_n) is a sequence of positive numbers with $\inf \{s_n \mid n \in \mathbb{N}\} = 0$, then there is a decreasing subsequence s_{n_k} that converges to 0 (see Inductive Construction 2 in the Subsequence Lecture)

(3) Check out Problem 1 in the Peyam 1 exam, and Problem 1 in the Peyam 2 exam, and also Problem 1 in the Peyam 4 exam
(4) Check out Problem 11.11

- Know the fact every sequence has a monotonic subsequence, but ignore its proof, although it is pretty neat!
- Know the Bolzano-Weierstraß Theorem, we used it sooo many times in this course!
- Know the fact that there is a subsequence of (s_n) that converges to $\limsup_{n\to\infty} s_n$. This makes lim sup a bit more concrete.
- Define: Limit point, and find limit points of sequences such as $s_n = (-1)^n$ or $s_n = \sin\left(\frac{\pi n}{3}\right)$ or $s_n = 0$ if n is even and n if n is odd. You can ignore the example with the enumeration of the rational numbers

- Know the following facts: If S is the set of limit points of (s_n), then
  (1) $S \neq \emptyset$ (since $\limsup s_n$ is in it)
  (2) $\sup(S) = \limsup s_n$ (ignore the proof)
  (3) (s_n) converges if and only if S is a single point (the proof is a nice application of the limsup squeeze theorem)
- Know that the set of limit points of (s_n) is closed. Don't memorize the proof, but check it out because it's a nice diagonal argument.
- Check out 11.8 and 11.9, as well as AP3 and AP4 in the section 11 homework

Section 12: lim sup and lim inf

- Of course this section asks you to know what lim inf and lim sup are ☺
- Show that, in general, $\limsup s_n t_n \neq (\limsup s_n)(\limsup t_n)$ (it's a good counterexample)
- Prove the limsup product rule: If $s_n \to s > 0$ then $\limsup s_n t_n = (\limsup s_n)(\limsup t_n)$, but only do it in the case where $\limsup t_n$ is finite
- I could ask you to do similar things for $\limsup s_n + t_n$, see AP1 and AP2 in the section 12 homework
- For the pre-ratio test, ignore its proof, but use it to show that if $\left|\frac{s_{n+1}}{s_n}\right| \to L$, then$|s_n|^{\frac{1}{n}} \to L$ and understand why it shows that the root test is better than the ratio test
- In the section 12 homework, you can ignore 12.4 (since I asked this on midterm 1) and 12.12 (too crazy), but check out 12.8, 12.14, AP1, and AP2

Section 14: Series

- Know the definition of a series and the definition of partial sums
- Define: A series converges, a series diverges
- You **don't** need to know facts like $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.
- Know the p series converges if and only if $p > 1$

- Show that if $a_n \geq 0$, then $\sum a_n$ converges if and only if (s_n) is bounded. The proof is a simple application of the Monotone Sequence Theorem

- You can ignore the derivation of the geometric series, but definitely know the result and that it converges if and only if $|r| < 1$. I could ask you for a specific example, like Example 6 in the Series lecture

- Define the Cauchy Criterion for a series. You don't need to know how to derive it

- Prove the Divergence test: If $\sum a_n$ converges, then $a_n \to 0$

- State and prove the Comparison Tests

- Define: $\sum a_n$ converges absolutely, and notice that absolute convergence $\Rightarrow$ convergence

- State the Root Test and know how to prove it in the case $\alpha < 1$. You don't need to know the proof for $\alpha = 1$ and $\alpha > 1$

- For the Ratio Test, I just want you to notice how it follows from the Root Test and the Pre-Ratio Test

- Understand that the root test is strictly better than the ratio test

- Of course, all the examples in the lecture and in the book (especially Examples $3 - 9$ in the book) are fair game for the exam, including those that I didn't cover in lecture

- In the section 14 homework, check out 14.6a, and check out AP3, AP4abc (with hints), AP6a. You can ignore AP5 and AP6b

Section 15: Alternating Series and Integral Tests

- Prove the Integral Test (both versions). I could ask you to prove it for a specific example, like Examples 1 and 2 in the book, or I could ask you to do it in general (like AP1 in the section 15 homework)
- As a corollary, show that the p series converges if and only if $p > 1$
- For the alternating series test: just know the statement and how to use it; ignore conditional convergence
- In the section 15 homework, check out problems 15.3 and 15.6 (great application of the Cauchy criterion), as well as AP4. In that problem you don't need to know the statement of the limit comparison test, but I could ask you to prove it. You don't need to know the Cauchy-Schwarz inequality, and you can ignore AP2, AP3, AP5, AP6, and AP8 (it's a good limsup exercise though)

Section 16: Decimal Expansions of Real Numbers

Ignore this section, it will not be on the exam

Section 17: Continuity

- Know both definitions of continuous functions (with sequences and $\epsilon-\delta$). Seriously, if you ever want to get a tattoo, you should choose the $\epsilon - \delta$ definition as your design (although it's a bit long, so it might hurt ☺). This definition will is over and over again in analysis.

- Show that the $\epsilon - \delta$ definition and the sequence definition are equivalent (Theorem 17.2), it's a very classical proof that uses ideas that we've talked about before
- Use both the $\epsilon - \delta$ definition and the sequence definition to show that some familiar functions are continuous:
  - ▶ Example 1: Basic Functions like $f(x) = x^2 + 1$
  - ▶ Example 2: $x^2 \sin\left(\frac{1}{x}\right)$ at $x = 0$ (using $\epsilon - \delta$ and sequences), $\frac{1}{x}\sin\left(\frac{1}{x^2}\right)$ (using sequences)
  - ▶ Example 3: Not continuous (see Lecture and also Problems 17.10ab)
  - ▶ Example 4: x^3 (Problem 17.9d)
  - ▶ Example 5: $|x|$ (AP1 in the section 17 homework)
  - ▶ Example 6: $\frac{1}{x}$ (AP1 in the section 17 homework)
  - ▶ Example 7: $\sqrt{x}$ (AP1 in the section 17 homework)
  - ▶ Lipschitz Functions (AP7 in the section 17 homework), you don't need to know the definition of Lipschitz, I would give it to you
- Show that if f and g are continuous, then $f + g$, kf, $f - g$, $|f|$, fg, $\frac{1}{f}$, $\frac{f}{g}$ are continuous. You need to know how to do that both with the sequence definition and the $\epsilon - \delta$ definition
- Show that $g \circ f$ is continuous, but just with the sequence definition

- You don't need to know the formula for $\max(f, g)$ in Example 8; I would give you the formula if needs be
- In the section 17 homework, check out problems 9, 10ab, 12, 13, AP7, and AP1, but ignore 8, 14, AP2, AP3

Section 18: Properties of Continuity

- Show that continuous functions on $[a, b]$ are bounded It's a very classical proof.
- In particular, notice how we use the idea of Contradiction + Bolzano-Weierstraß . This idea appears over and over again in analysis
- Prove the Extreme Value Theorem. I know it looks long, but the main idea is to find a sequence that converges to the sup of f
- Prove the Intermediate Value Theorem. The main idea is to show that there is x such that $f(x) \leq c$ and $f(x) \geq c$
- Know the statement that if f is continuous and I is an interval, then $f(I)$ is an interval, but ignore its proof
- Know how to show some of the applications of the Intermediate Value Theorem: For example, if $f : [0, 1] \to [0, 1]$ is continuous, then f has at least one fixed point (Example 1), or that for every $y \geq 0$ there is some $x \geq 0$ such that $x^2 = y$ (Example 2)
- Know the fact that one-to-one continuous functions are either strictly increasing or strictly decreasing, but ignore its proof

- Know that if f is one-to-one and continuous, then f^{-1} is continuous, but ignore its proof
- In the section 18 homework, know how to do 18.9 and 18.10, but ignore 18.12 and AP4
- Check out Problems 6 and 7 in the Peyam 4 exam

Section 19: Uniform Continuity

- The only thing I ask you in this section is to state the definition of uniform continuity and to show from the definition that f is uniformly continuous.
- Know the definition of uniform continuity, and understand how it's different from the definition of continuity. The main point is that, in uniform continuity, the δ does not depend on x and y, it's independent of where you are
- Use the $\epsilon - \delta$ definition of uniform continuity to show that f is uniformly continuous. For example, show that $f(x) = x^2$ is uniformly continuous on $[-1, 3]$ (Example 1) or that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[2, \infty)$ (Example 2)
- I won't ask you to show that f is not uniformly continuous, and you don't need to know *yet* that if f is continuous on $[a, b]$, then it is uniformly continuous (that will be part of the final)
- If you want some practice problems, check out 19.1(b) and 19.2