SOLUTIONS

MOCK MIDTERM

Instructions: This is a mock midterm, designed to give you some practice for the actual midterm. It will be similar in length and in difficulty to the actual midterm, but beware that the actual exam might have different questions! So please also look at the study guide and the suggested homework for a more complete study experience!

1		20
2		20
3		20
4		20
5		20
Total		100

Note: The equations for spherical coordinates are $x = \rho \sin(\phi)\cos(\theta)$, $y = \rho \sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$ and the Jacobian is $\rho^2 \sin(\phi)$.

ANSWERS

1. $\frac{79}{60}$

2. 8π

3. $\frac{3\pi}{4}$

4. $\frac{1}{5}\left(3^{\frac{5}{2}} - 1\right)$

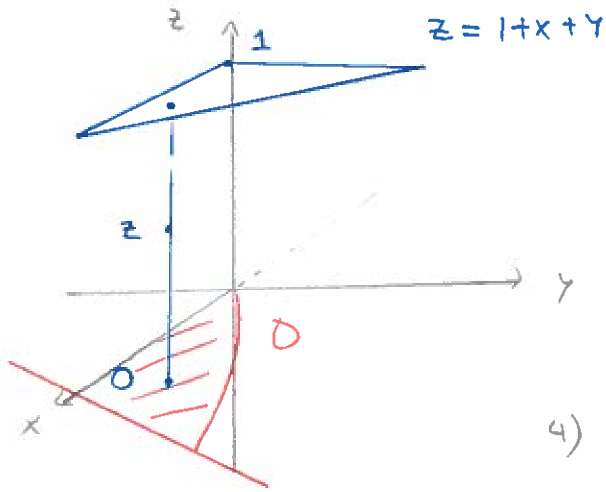
Date: Friday, November 2, 2018.

5. (a) 0

   (b) 0

1. *(20 points)* Calculate the volume of the region E under the plane $z = 1 + x + y$ and above the region in the xy–plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

1) PICTURE

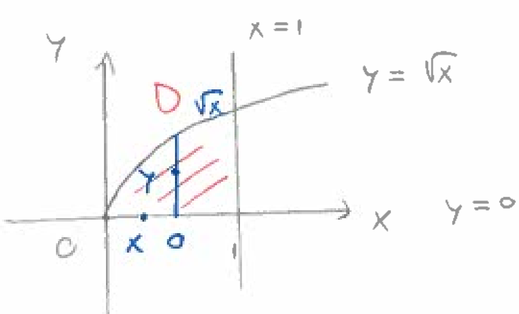


2) $V = \iiint_E 1 \, dx\, dy\, dz$

small $\leq z \leq$ big

$0 \leq z \leq 1+x+y$

3) FIND D



small $\leq y \leq$ big

$0 \leq y \leq \sqrt{x}$

$0 \leq x \leq 1$

4) $V = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 1 \, dz\, dy\, dx$

$= \int_0^1 \int_0^{\sqrt{x}} 1 + x + y \, dy\, dx$

$= \int_0^1 \left[y + xy + \frac{y^2}{2} \right]_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 \underbrace{\sqrt{x}}_{x^{\frac{1}{2}}} + \underbrace{x\sqrt{x}}_{x^{\frac{3}{2}}} + \frac{x}{2} \, dx$

$= \left[\frac{2}{3} x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}} + \frac{x^2}{4} \right]_0^1$

$= \frac{2}{3} + \frac{2}{5} + \frac{1}{4}$

$= \frac{79}{60}$

SPHERICAL

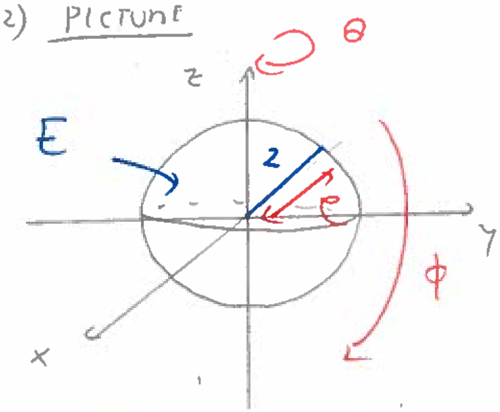
2. *(20 points)* Recall that the mass of a solid E with density function f is $\int\int\int_E f(x, y, z)dxdydz$.

Suppose that a black hole is a ball centered at the origin and radius 2, and has density $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$. Calculate the mass of the black hole.

1) $\text{Mass} = \iiint_E f(x,y,z)\, dx\,dy\,dz$

$$= \iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\, dx\,dy\,dz$$

2) PICTURE



$0 \leq \rho \leq 2$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

3) $$\iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\, dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{2} \frac{1}{\rho}\, \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

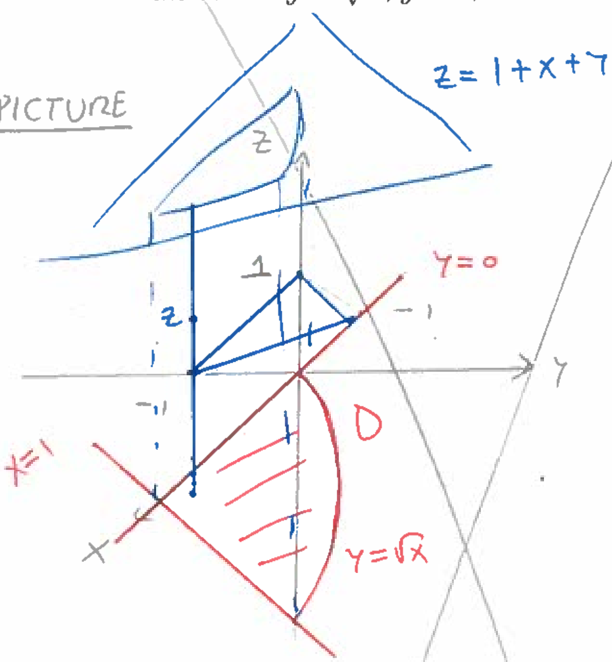
$$= \left(\int_0^2 \rho\, d\rho\right)\left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^{\pi} \sin(\phi)\, d\phi\right)$$

$$= \left[\tfrac{1}{2}\rho^2\right]_0^2 \left[\theta\right]_0^{2\pi} \left[-\cos(\phi)\right]_0^{\pi}$$

$$= (2)(2\pi)(2) = 8\pi$$

1. *(20 points)* Calculate the volume of the region E under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

1) PICTURE



INTERCEPTS

z-INT $(x=0, y=0)$ $z = 1+0+0 = 1 \Rightarrow z=1$

x-INT $(y=0, z=0)$ $0 = 1+x+0 \Rightarrow x=-1$

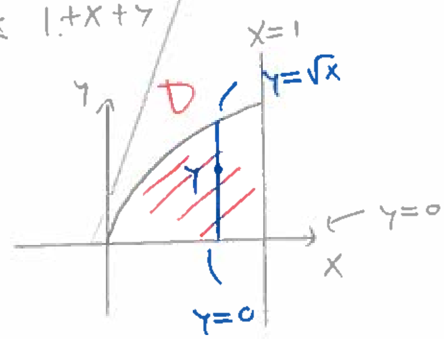
y-INT $(x=0, z=0)$ $0 = 1+0+y \Rightarrow y=-1$

2) $V = \iiint_E 1 \, dx\,dy\,dz$

SMALL $\leq z \leq$ BIG

$0 \leq z \leq 1+x+y$

3) FIND D



SMALL $\leq y \leq$ BIG

$0 \leq y \leq \sqrt{x}$

$0 \leq x \leq 1$

4) $V = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1-x-y} 1 \, dz\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} 1-x-y \, dy\,dx$

$= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 \sqrt{x} - x\sqrt{x} - \frac{x}{2} \, dx = \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} - \frac{x^2}{2} \right]_0^1$

$= \frac{2}{3} - \frac{2}{5} - \frac{1}{2}$

$= \frac{20}{30} - \frac{12}{30} - \frac{15}{30}$

$= -\frac{3}{30} = -\frac{1}{10}$

CYLINDRICAL

3. *(20 points)* Find $\int\int\int_E z\,dx\,dy\,dz$, where E is the solid between the surfaces

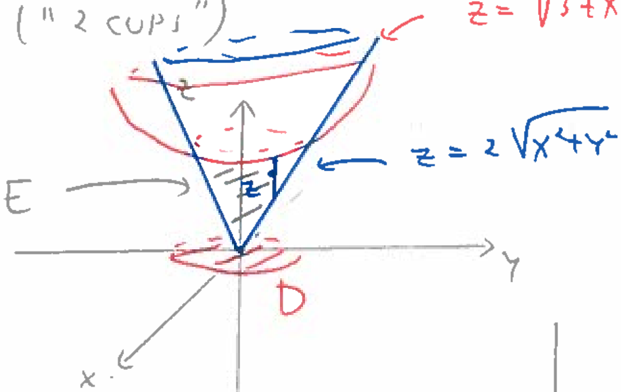
$$z = \sqrt{3+x^2+y^2} \text{ and } z = 2\sqrt{x^2+y^2}$$

$\Rightarrow z^2 = 3+x^2+y^2$ $\qquad \Rightarrow \underbrace{z^2 = 4(x^2+y^2)}_{\text{CONE}}$

$\Rightarrow \underbrace{z^2 - x^2 - y^2 = 3}$

HYPERBOLOID OF 2 SHEETS
("2 CUPS")

1) PICTURE



2) INTERSECTION

$\sqrt{3+x^2+y^2} = 2\sqrt{x^2+y^2}$

$3+x^2+y^2 = 4(x^2+y^2)$

$3+x^2+y^2 = 4x^2+4y^2$

$\Rightarrow 3x^2+3y^2 = 3$

$\Rightarrow x^2+y^2 = 1$

$\Rightarrow D =$ DISK OF RADIUS 1

3) $0 \le r \le 1$

$0 \le \theta \le 2\pi$

$2\sqrt{x^2+y^2} \le z \le \sqrt{3+x^2+y^2}$

$2r \le z \le \sqrt{3+r^2}$

4) ANS

$$= \int_0^{2\pi}\int_0^1\int_{2r}^{\sqrt{3+r^2}} z r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[\frac{z^2}{2} r\right]_{z=2r}^{z=\sqrt{3+r^2}} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{(3+r^2)}{2} r - \frac{4r^2}{2} r \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{3}{2} r + \frac{1}{2} r^3 - 2r^3 \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{3}{2} r - \frac{3}{2} r^3 \, dr\, d\theta$$

$$= 2\pi \left[\frac{3}{4} r^2 - \frac{3}{8} r^4\right]_0^1$$

$$= 2\pi\left(\frac{3}{4} - \frac{3}{8}\right)$$

$$= 2\pi\left(\frac{3}{8}\right) = \frac{3\pi}{4}$$

IN THE FIRST QUADRANT

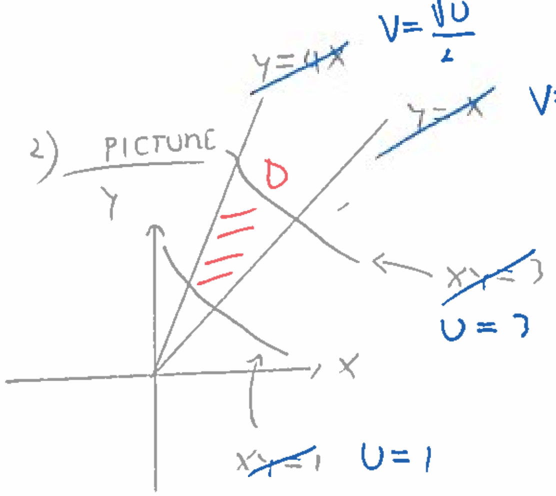
4. *(20 points)* Evaluate $\int\int_D x^2y\,dx\,dy$, where D is the region between the lines $y = x, y = 4x$ and the hyperbolas $xy = 1, xy = 3$.

$y=\frac{1}{x}$ $\quad y=\frac{3}{x}$

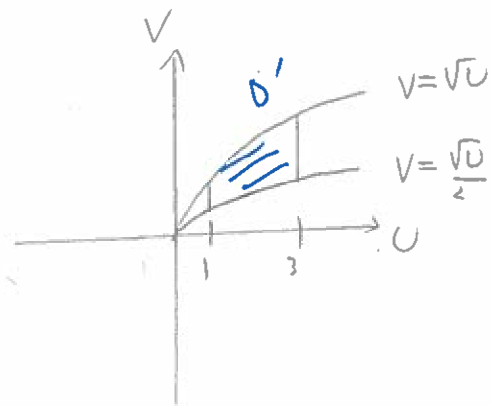
1) LET $U = XY$

$V = X$

2) PICTURE



U, V



$xy=1 \Rightarrow \underline{U=1}$

$xy=3 \Rightarrow \underline{U=3}$

$V=x$

$y=x \Rightarrow \frac{U}{x}=x \Rightarrow U=x^2 \Rightarrow U=V^2 \Rightarrow \underline{V=\sqrt{U}}$

$U=XY$

$y=\frac{U}{x}$

$y=4x \Rightarrow \frac{U}{x}=4x \Rightarrow U=4x^2 \Rightarrow U=4V^2 \Rightarrow V^2=\frac{U}{4} \Rightarrow V=\sqrt{\frac{U}{4}}=\underline{\frac{\sqrt{U}}{2}}$

so D': $1 \le U \le 3$

$\frac{\sqrt{U}}{2} \le V \le \sqrt{U}$

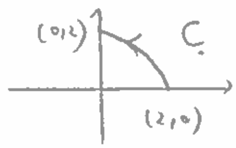
$x \geq 0$ $\quad V=x$

3) $dUdV = \left|\frac{dUdV}{dxdy}\right| dx\,dy = |-x|\,dxdy = x\,dxdy \Rightarrow dxdy = \frac{1}{x}\,dUdV = \frac{1}{V}\,dUd$

$$\frac{dUdV}{dxdy} = \begin{vmatrix} \partial U/\partial x & \partial U/\partial y \\ \partial V/\partial x & \partial V/\partial y \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x$$

5. *(20 points)* Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = xy^2\,\mathbf{i} + x^2y\,\mathbf{j}$ and C is the arc of circle $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$:

(a) *(10 points)* Using the definition of the line integral

1) 

2) $x(t) = 2\cos(t)$ $\quad 0 \le t \le \pi/2$

$y(t) = 2\sin(t)$

3) $$\int_C F \cdot dr = \int_0^{\pi/2} \langle x(t)(y(t))^2, (x(t))^2 y(t) \rangle \cdot \langle x'(t), y'(t) \rangle \, dt$$

$$= \int_0^{\pi/2} \langle 8\cos(t)\sin^2(t), 8\cos^2(t)\sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt$$

$$= \int_0^{\pi/2} -8\cos(t)\sin^3(t) + 8\cos^3(t)\sin(t) \, dt$$

(b) *(10 points)* Using the fundamental theorem for line integrals

$$= \left[-2\sin^4(t) - 2\cos^4(t) \right]_0^{\pi/2}$$

$$= -2 + 0 + 0 + 2$$

$$= 0$$

(a) FIND f SUCH THAT $\nabla f = F \Rightarrow \langle f_x, f_y \rangle = \langle xy^2, x^2y \rangle$

$f_x = xy^2 \Rightarrow f(x,y) = \frac{1}{2}x^2y^2 + \text{JUNK}$

$f_y = x^2y \Rightarrow f(x,y) = \frac{1}{2}x^2y^2 + \text{JUNK}$ $\Rightarrow f(x,y) = \frac{1}{2}x^2y^2$

$$\int_C F \cdot dr = \int_C \nabla f(r) \cdot dr = \overset{\text{END}}{f(0,2)} - \overset{\text{START}}{f(2,0)}$$

$$= \frac{1}{2}0^2 2^2 - \frac{1}{2}2^2 0^2 = 0$$

FINALLY, $x^2 y \underset{u=xy}{=} v^2\left(\frac{u}{x}\right) \underset{v=x}{=} v^2\left(\frac{u}{v}\right) = uv$

so $$\iint_D x^2 y \, dx\, dy = \iint_{D'} u\cancel{v}\left(\frac{1}{\cancel{v}}\right) du\, dv$$

$$= \int_1^3 \int_{\frac{\sqrt{u}}{2}}^{\sqrt{u}} u \, dv\, du$$

$$= \int_1^3 \left[uv\right]_{v=\frac{\sqrt{u}}{2}}^{v=\sqrt{u}} du$$

$$= \int_1^3 u\sqrt{u} - \frac{u\sqrt{u}}{2} \, du$$

$$= \int_1^3 \frac{1}{2} u\sqrt{u}\, du$$

$$= \int_1^2 \frac{1}{2} u^{\frac{3}{2}} du$$

$$= \left[\frac{1}{\cancel{2}} \frac{\cancel{2}}{5} u^{\frac{5}{2}}\right]_1^3$$

$$= \boxed{\frac{1}{5}\left(3^{\frac{5}{2}} - 1\right)}$$

$$\left(= \frac{1}{5}\left(9\sqrt{3} - 1\right)\right.$$

Here's a (perhaps) easier solution to Problem 4:

Problem 4: Notice here the region is $\frac{y}{x} = 1$, $\frac{y}{x} = 4$, $xy = 1$, $xy = 3$.

STEP 1:

$$\begin{cases} u = & \dfrac{y}{x} \\ v = & xy \end{cases}$$

STEP 2: Find D'

In terms of u and v, our new region becomes

$$\begin{cases} 1 \leq u \leq 4 \\ 1 \leq v \leq 3 \end{cases}$$

So D' is a rectangle with sides $[1, 4] \times [1, 3]$

STEP 3: Jacobian:

$$dudv = \left| \frac{dudv}{dxdy} \right| dxdy = \left| -2 \left(\frac{y}{x} \right) \right| dxdy = 2 \left(\frac{y}{x} \right) dxdy = 2udxdy$$

$$\frac{dudv}{dxdy} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = \left(-\frac{y}{x^2} \right) x - \left(\frac{1}{x} \right) y = -\frac{y}{x} - \frac{y}{x} = -2 \left(\frac{y}{x} \right)$$

Therefore $dxdy = \frac{1}{2u} dudv$

STEP 4: Integrate:

Function: Notice here

$$uv = \left(\frac{y}{x}\right) xy = y^2 \Rightarrow y = \sqrt{uv}$$
$$\frac{u}{v} = \frac{\frac{y}{x}}{xy} = \frac{1}{x^2} \Rightarrow x^2 = \frac{v}{u}$$

Therefore:

$$f(x,y) = x^2 y = \left(\frac{v}{u}\right)\sqrt{uv} = \frac{v\sqrt{v}}{\sqrt{u}}$$

$$\begin{aligned}
\int\int_D x^2 y \, dxdy &= \int\int_{D'} \left(\frac{v\sqrt{v}}{\sqrt{u}}\right)\left(\frac{1}{2u}\right) dudv \\
&= \frac{1}{2}\int_1^3\int_1^4 v\sqrt{v}\left(\frac{1}{u\sqrt{u}}\right) dudv \\
&= \frac{1}{2}\left(\int_1^4 u^{-\frac{3}{2}} du\right)\left(\int_1^3 v^{\frac{3}{2}} dv\right) \\
&= \frac{1}{2}\left[-2u^{-\frac{1}{2}}\right]_1^4 \left[\frac{2}{5}v^{\frac{5}{2}}\right]_1^3 \\
&= -\frac{2}{5}\left(\frac{1}{\sqrt{4}} - 1\right)\frac{2}{3}\left(3^{\frac{5}{2}} - 1^{\frac{5}{2}}\right) \\
&= -\frac{2}{5}\left(-\frac{1}{2}\right)\left(3^{\frac{5}{2}} - 1\right) \\
&= \frac{1}{5}\left(3^{\frac{5}{2}} - 1\right)
\end{aligned}$$