MIDTERM 3 – REVIEW

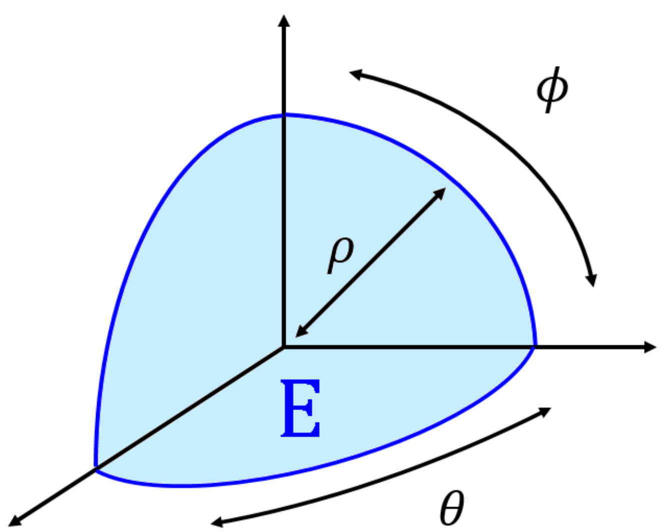
1. Spherical Coordinates

Example 1:

$$\iiint_E x^2 + y^2 dxdydz$$

E is the ball of radius 2 in the first octant

STEP 1: Picture



STEP 2: Inequalities

Date: Tuesday, November 9, 2021.

$$\begin{cases} 0 \leq \rho \leq 2 \\ 0 \leq \theta \leq \dfrac{\pi}{2} \\ 0 \leq \phi \leq \dfrac{\pi}{2} \end{cases}$$

STEP 3: **Integrate**

Function: $f(x,y,z) = x^2 + y^2 = r^2 = (\rho \sin(\phi))^2 = \rho^2 \sin^2(\phi)$

$$\begin{aligned} &\iiint_E x^2 + y^2 \, dxdydz \\ &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) d\rho d\theta d\phi \\ &= \left(\frac{\pi}{2}\right) \left(\int_0^2 \rho^4 d\rho\right) \left(\int_0^{\frac{\pi}{2}} \sin^3(\phi) d\phi\right) \end{aligned}$$

$$\int_0^2 \rho^4 d\rho = \left[\frac{\rho^5}{5}\right]_0^2 = \frac{2^5}{5} = \frac{32}{5}$$

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \sin^3(\phi) d\phi \\ &= \int_0^{\frac{\pi}{2}} \sin^2(\phi) \sin(\phi) d\phi \\ &= \int_0^{\frac{\pi}{2}} \left(1 - \cos^2(\phi)\right) \sin(\phi) d\phi \end{aligned}$$

$$(u = \cos(\phi), du = -\sin(\phi)d\phi \Rightarrow \sin(\phi) = -du,$$
$$u(0) = \cos(0) = 1, u\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0)$$
$$= \int_1^0 \left(1 - u^2\right)(-du)$$
$$= \int_0^1 1 - u^2 du$$
$$= \left[u - \frac{u^3}{3}\right]_0^1$$
$$= \left(1 - \frac{1}{3}\right)$$
$$= \frac{2}{3}$$

$$\text{Answer: } \frac{\pi}{2} \times \frac{32}{5} \times \frac{2}{3} = \frac{32\pi}{15}$$

2. Integral over SIMS

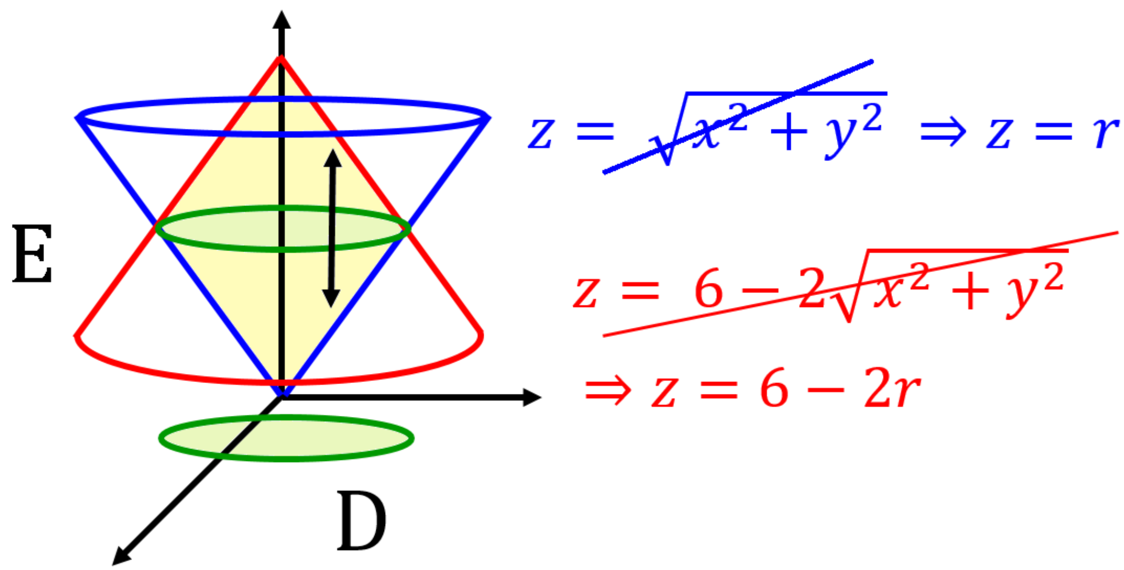
Video: Integral over SIMS

Example 2:

Find the volume of the region E enclosed by the surfaces

$$z = \sqrt{x^2 + y^2} \text{ and } z = 6 - 2\sqrt{x^2 + y^2}$$

STEP 1: Picture



STEP 2: Inequalities:

$$\begin{aligned} \text{Small } \leq & z \leq \text{ Big} \\ r \leq & z \leq 6 - 2r \end{aligned}$$

STEP 3: Find D: Intersection:

$$\begin{aligned} r =& 6 - 2r \\ 3r =& 6 \\ r =& 2 \end{aligned}$$

Hence D is a disk of radius 2, so

$$\begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \\ r \leq z \leq 6 - 2r \end{cases}$$

STEP 3: Integrate:

$$\begin{aligned}
\text{Vol}(E) &= \iiint_E 1\,dxdydz \\
&= \int_0^{2\pi} \int_0^2 \int_r^{6-2r} r\,dzdrd\theta \\
&= 2\pi \int_0^2 r\,(6-2r-r)\,dr \\
&= 2\pi \int_0^2 6r - 3r^2 dr \\
&= 2\pi \left[3r^2 - r^3\right]_0^2 \\
&= 2\pi \left(3(2)^2 - 2^3\right) \\
&= 2\pi\,(12-8) \\
&= 2\pi(4) \\
&= 8\pi
\end{aligned}$$

3. The Jacobian

Video: Change of Variables

Example 3:

Use the following change of variables to calculate

$$\iint_D x^2 + xy + y^2 dxdy$$

Where D is the region enclosed by the ellipse $x^2 + xy + y^2 = 3$

STEP 1: (here will be given)

$$\begin{cases} x = & u + v\sqrt{3} \\ y = & u - v\sqrt{3} \end{cases}$$

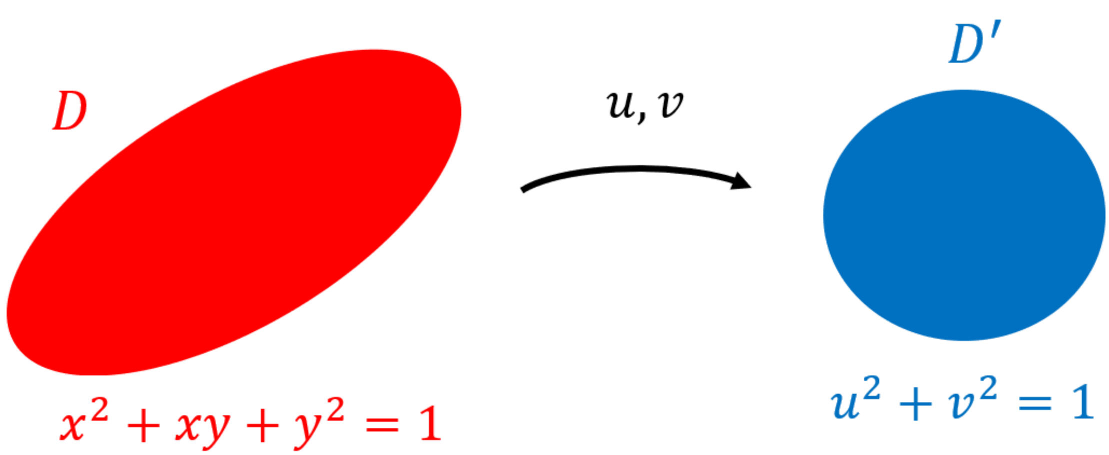
STEP 2: Endpoints

Let's see what happens to D:

$$\begin{aligned} x^2 + xy + y^2 &= 3 \\ \left(u + v\sqrt{3}\right)^2 + \left(u + v\sqrt{3}\right)\left(u - v\sqrt{3}\right) + \left(u - v\sqrt{3}\right)^2 &= 3 \\ u^2 + \cancel{2uv\sqrt{3}} + 3v^2 + u^2 - \cancel{2uv\sqrt{3}} + \cancel{3v^2} + u^2 - \cancel{3v^2} &= 3 \\ 3u^2 + 3v^2 &= 3 \\ u^2 + v^2 &= 1 \end{aligned}$$

So D becomes a disk D' of radius 1

$$\begin{cases} 0 \leq & r \leq 1 \\ 0 \leq & \theta \leq 2\pi \end{cases}$$



STEP 3: Jacobian:

$$dxdy = \left|\frac{dxdy}{dudv}\right| dudv = \left|-2\sqrt{3}\right| dudv = 2\sqrt{3}dudv$$

$$\frac{dxdy}{dudv} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & \sqrt{3} \\ 1 & -\sqrt{3} \end{vmatrix} = -\sqrt{3} - \sqrt{3} = -2\sqrt{3}$$

STEP 4: Integrate:

$$f(x, y) = x^2 + xy + y^2 = 3u^2 + 3v^2$$

$$\begin{aligned}
&\iint_D x^2 + xy + y^2 \, dxdy \\
=&\iint_{D'} \left(3u^2 + 3v^2\right) 2\sqrt{3}\, dudv \\
=&2\sqrt{3} \int_0^{2\pi} \int_0^1 3r^2 r dr d\theta \\
=&2\sqrt{3}(2\pi)3 \int_0^1 r^3 dr \\
=&12\sqrt{3}\left(\frac{1}{4}\right)\pi \\
=&3\sqrt{3}\pi
\end{aligned}$$

4. Triple Integrals

Example 4:

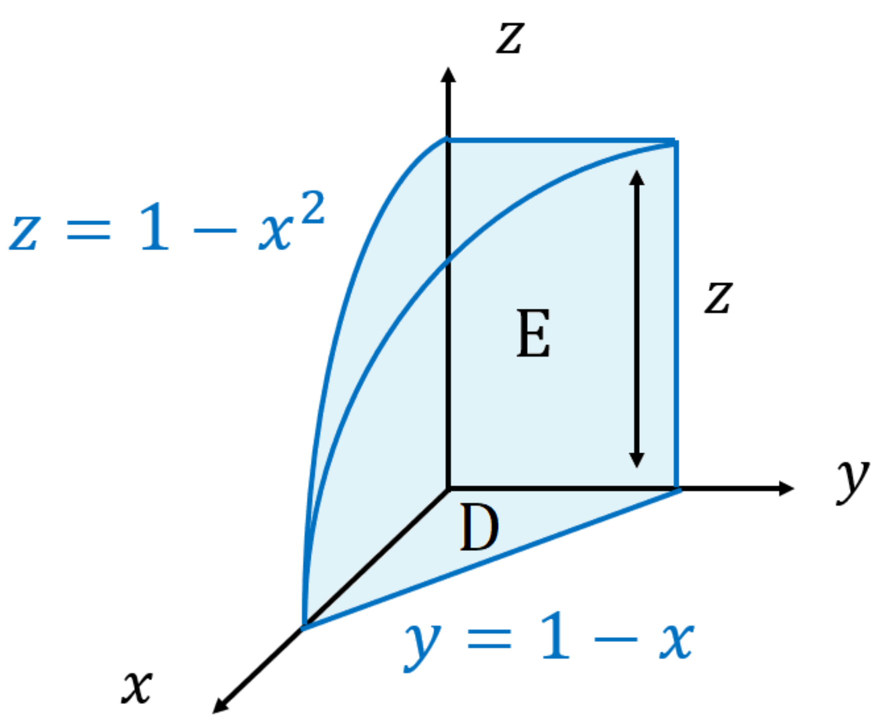
Set up, but do **not** evaluate the following triple integral. Write it in the usual form with $dzdydx$.

$$\iiint_E \sin(y)\, dxdydz$$

Here E is the solid in the first octant bounded by the surfaces $z = 1 - x^2$ and $y = 1 - x$.

STEP 1: Picture

Here $z = 1 - x^2$ is a cylinder in the $y-$direction and $y = 1 - x$ is a plane/cylinder in the z-direction

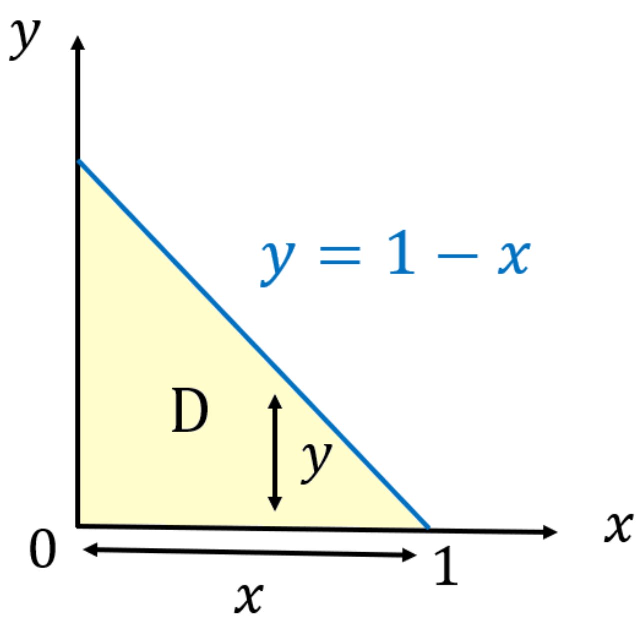


Note: Usually it's totally fine to do this in the $y-$direction, but here we specify $dzdydx$, which means you have to do z first.

STEP 2: Inequalities

$$\text{Small } \leq z \leq \text{ Big}$$
$$0 \leq z \leq 1 - x^2$$

Find D:



$$0 \leq y \leq 1 - x$$
$$0 \leq x \leq 1$$

Therefore our inequalities are:

$$\begin{cases} 0 \leq z \leq 1 - x^2 \\ 0 \leq y \leq 1 - x \\ 0 \leq x \leq 1 \end{cases}$$

STEP 3: Answer:

$$\iiint_E \sin(y)\, dxdydz = \int_0^1 \int_0^{1-x} \int_0^{1-x^2} \sin(y)\, dzdydx$$