MAT 267 – MIDTERM 3 – SOLUTIONS

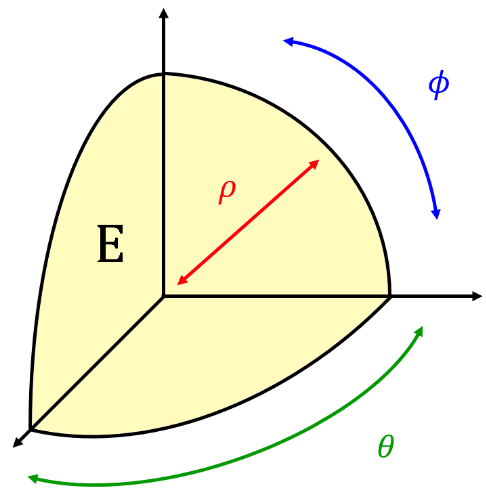
1. Multiple Choice

(1) **D**

$$
\begin{aligned}
&\int_0^1 \int_0^2 \int_0^3 8xyz \, dxdydz \\
=&8\left(\int_0^3 xdx\right)\left(\int_0^2 ydy\right)\left(\int_0^1 zdz\right) \\
=&\cancel{8}\left[\frac{x^2}{\cancel{2}}\right]_0^3 \left[\frac{y^2}{\cancel{2}}\right]_0^2 \left[\frac{z^2}{\cancel{2}}\right]_0^1 \\
=&\left(2^2\right)\left(3^2\right)\left(1^2\right) \\
=&4 \times 9 \\
=&36
\end{aligned}
$$

(2) **B**

STEP 1: Picture:



STEP 2: Inequalities:

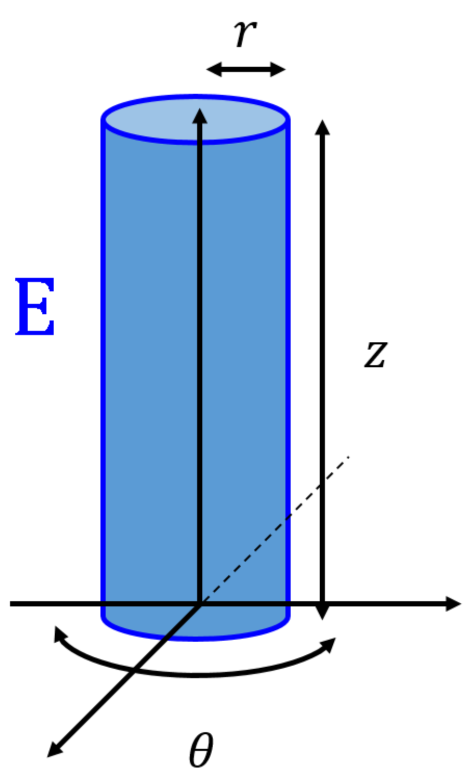
$$0 \leq \rho \leq 1$$
$$0 \leq \theta \leq \frac{\pi}{2}$$
$$0 \leq \phi \leq \frac{\pi}{2}$$

STEP 3: Integrate: $f(x,y,z) = \rho$, so

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho \, \rho^2 \sin(\phi) \, d\rho d\theta d\phi = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \sin(\phi) \, d\rho d\theta d\phi$$

(3) **C**

STEP 1: Picture:



STEP 2: Inequalities:

$$0 \le r \le R$$
$$0 \le \theta \le 2\pi$$
$$0 \le z \le h$$

STEP 3: Integrate: Here

$$f(x, y, z) = \underbrace{x^2 + y^2}_{r^2} + z^2 = r^2 + z^2$$

$$\int_0^{2\pi} \int_0^R \int_0^h \left(r^2 + z^2\right) r \, dz dr d\theta = \int_0^{2\pi} \int_0^R \int_0^h r^3 + rz^2 \, dz dr d\theta$$

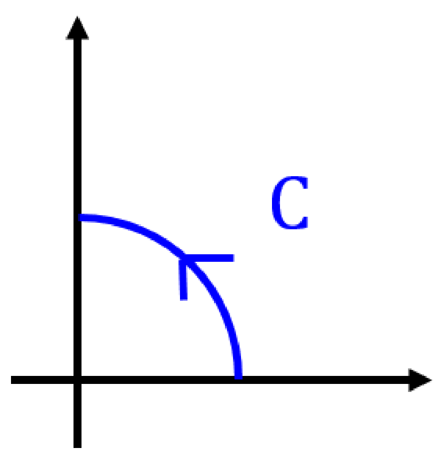
(4) **D**

For each vector field, we need to check whether $P_y = Q_x$ or not:

(A) $P_y = -2y$, $Q_x = 1$, No
(B) $P_y = x^3, Q_x = y^3$, No
(C) $P_y = 2, Q_x = 0$, No
(D) $P_y = -4, Q_x = -4$, Bingo!

(5) **C**

STEP 1: Picture:



STEP 2: Parametrize C

$$\begin{aligned} x(t) =& 3\cos(t) \\ y(t) =& 3\sin(t) \\ 0 \leq & t \leq 2\pi \end{aligned}$$

STEP 3:

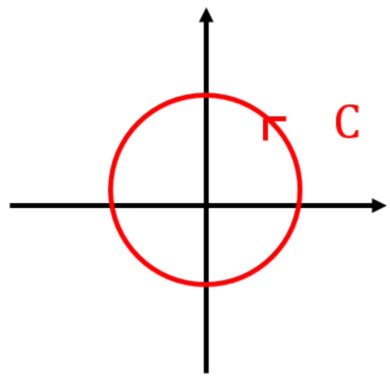
$$\begin{aligned}
ds =& \sqrt{(x'(t))^2 + (y'(t))^2}dt \\
=& \sqrt{(-3\sin(t))^2 + (3\cos(t))^2}dt \\
=& \sqrt{9\sin^2(t) + 9\cos^2(t)}dt \\
=& \sqrt{9}dt \\
=& 3dt
\end{aligned}$$

STEP 4: Integrate

$$\begin{aligned}
\int_C x ds =& \int_0^{\frac{\pi}{2}} 3\cos(t) 3 dt \\
=& \int_0^{\frac{\pi}{2}} 9\cos(t) dt \\
=& [9\sin(t)]_0^{\frac{\pi}{2}} \\
=& 9\sin\left(\frac{\pi}{2}\right) - \sin(0) \\
=& 9
\end{aligned}$$

(6) **C**

STEP 1: Picture



STEP 2: **Check Conservative**

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \left(2x^3\right)_x - \left(-2y^3\right)_y = 6x^2 + 6y^2 \neq 0$$

STEP 3: **Integrate**

$$\begin{aligned}
\int_C F \cdot dr &= \int \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy \\
&= \int \int_D 6x^2 + 6y^2 dxdy \\
&= \int_0^{2\pi} \int_0^1 6r^2\, r\, drd\theta \\
&=2\pi \int_0^1 6r^3 dr \\
&=2\pi \left[\frac{6}{4} r^4\right]_0^1 \\
&=2\pi \left(\frac{3}{2}\right) \\
&=3\pi
\end{aligned}$$

(7) **D**

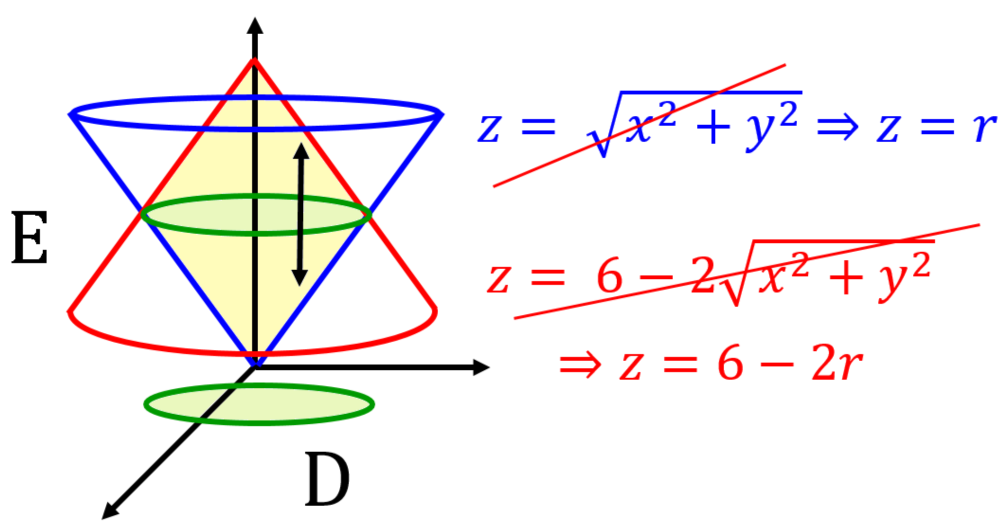
$$F = \nabla f$$
$$\langle 2x, 4y, 8z\rangle = \langle f_x, f_y, f_z\rangle$$

$$f_x = 2x \Rightarrow f = \int 2x\,dx = x^2 + \text{ JUNK}$$
$$f_y = 4y \Rightarrow f = \int 4y\,dy = 2y^2 + \text{ JUNK}$$
$$f_z = 8z \Rightarrow f = \int 8z\,dz = 4z^2 + \text{ JUNK}$$

Therefore a potential function is $f(x, y, z) = x^2 + 2y^2 + 4z^2$

2. Free Response

1. STEP 1: Picture



Doesn't that picture remind you of the Sims game? ☺

STEP 2: Inequalities:

$$\begin{aligned} \text{Small } &\leq z \leq \text{ Big} \\ \sqrt{x^2+y^2} &\leq z \leq 6 - 2\sqrt{x^2+y^2} \\ r &\leq z \leq 6 - 2r \end{aligned}$$

STEP 3: Find D: Intersection:

$$\begin{aligned} \sqrt{x^2+y^2} &= 6 - 2\sqrt{x^2+y^2} \\ r &= 6 - 2r \\ 3r &= 6 \\ r &= 2 \end{aligned}$$

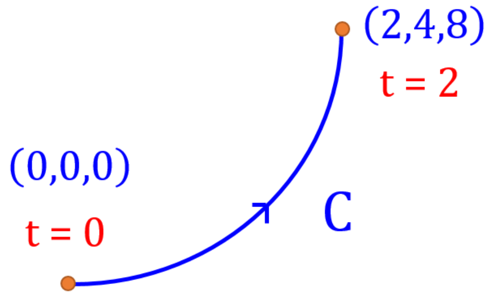
Hence D is a disk of radius 2, so

$$\begin{aligned} 0 \leq r &\leq 2 \\ 0 \leq \theta &\leq 2\pi \\ r \leq z &\leq 6 - 2r \end{aligned}$$

STEP 3: Integrate:

$$
\begin{aligned}
&\text{Vol(E)}\\
=&\int\int\int_E 1\, dxdydz\\
=&\int_0^{2\pi}\int_0^2\int_r^{6-2r} r\, dzdrd\theta\\
=&2\pi\int_0^2 r\,(6-2r-r)\,dr\\
=&2\pi\int_0^2 6r-3r^2dr\\
=&2\pi\left[3r^2-r^3\right]_0^2\\
=&2\pi\left(3(2)^2-2^3\right)\\
=&2\pi\,(12-8)\\
=&2\pi(4)\\
=&8\pi
\end{aligned}
$$

2. STEP 1: Picture:



Note: $(0,0,0)$ corresponds to $\langle t,t^2,t^3\rangle = \langle 0,0,0\rangle$ so $t=0$ and $(2,4,8)$ corresponds to $\langle t,t^2,t^3\rangle = \langle 2,4,8\rangle$ so $t=2$.

STEP 2: Integrate:

$$\begin{aligned}
&\int_C F \cdot dr \\
=&\int_0^2 F(r(t)) \cdot r'(t)dt \\
=&\int_0^2 \langle -3y(t), 3x(t), 2z(t)\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\, dt \\
=&\int_0^2 \left\langle -3t^2, 3t, 2t^3\right\rangle \cdot \left\langle 1, 2t, 3t^2\right\rangle dt \\
=&\int_0^2 -3t^2 + 3t\,(2t) + 2t^3\left(3t^2\right) dt \\
=&\int_0^2 -3t^2 + 6t^2 + 6t^5 dt \\
=&\int_0^2 3t^2 + 6t^5 dt \\
=&\left[t^3 + t^6\right]_0^2 \\
=&2^3 + 2^6 \\
=&8 + 64 \\
=&72
\end{aligned}$$

3a.

$$P_y = (4x + 3e^y)_y = 3e^y$$
$$Q_x = \left(3xe^y + 3y^2\right)_x = 3e^y$$

$P_y = Q_x$, so F is conservative

3b.

$$F = \nabla f$$
$$\left\langle 4x + 3e^y, 3xe^y + 3y^2\right\rangle = \langle f_x, f_y\rangle$$

$$
\begin{aligned}
f_x =& 4x + 3e^y \Rightarrow f = \int 4x + 3e^y dx = 2x^2 + 3xe^y + \text{ JUNK} \\
f_y =& 3xe^y + 3y^2 \Rightarrow f = \int 3xe^y + 3y^2 dy = 3xe^y + y^3 + \text{ JUNK}
\end{aligned}
$$

Hence $f(x, y) = 2x^2 + 3xe^y + y^3$

3c.

$$
\begin{aligned}
\int_C F \cdot dr =& \int_C \nabla f \cdot dr \\
=& f(1, 2) - f(3, 0) \\
=& 2(1)^2 + 3(1)e^2 + 2^3 - 2(3)^2 - 3(3)e^0 - 0^3 \\
=& 2 + 3e^2 + 8 - 18 - 9 \\
=& 3e^2 - 17
\end{aligned}
$$