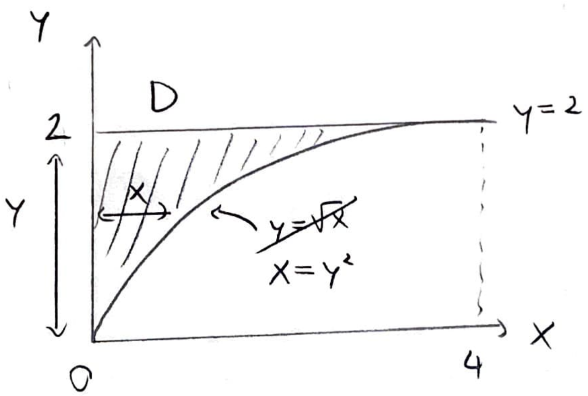
SOLUTIONS

1. *(10 points)* Evaluate the following *impossible* integral by changing the order of integration

$$\int_0^4 \int_{\sqrt{x}}^2 10\, x \cos\left(y^5\right)\, dy dx$$

1) INEQUALITIES $\begin{cases} \sqrt{x} \leq y \leq 2 \\ 0 \leq x \leq 4 \end{cases}$

2) PICTURE



3) INEQUALITIES (AGAIN)

$$\begin{cases} 0 \leq x \leq y^2 \\ 0 \leq y \leq 2 \end{cases}$$

4) INTEGRATE

$$\int_0^2 \int_{\sqrt{x}}^2 10x \cos(y^5)\, dy\, dx = \int_0^2 \int_0^{y^2} 10x \cos(y^5)\, dx\, dy$$

Answer	$\sin(32)$

$$= \int_0^2 \left[5x^2 \cos(y^5)\right]_{x=0}^{x=y^2} dy$$

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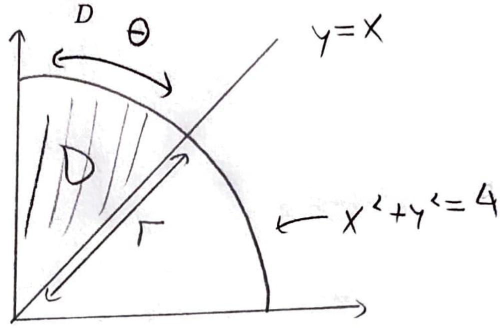
$$= \int_0^2 5y^4 \cos(y^5)\, dy = \left[\sin(y^5)\right]_0^2 = \sin(2^5) = \sin(32)$$

2. *(10 points)* Evaluate the following integral, where D is the region in the first quadrant bounded by the disk $x^2 + y^2 = 4$, the line $y = x$, and the $y-$axis

$$\iint_D 2y^2\, dx dy$$

1) PICTURE



2) INEQUALITIES

$$\begin{cases} 0 \leq r \leq 2 \\ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \end{cases}$$

3) INTEGRATE

$$\iint_D 2y^2\, dx dy = \int_{\pi/4}^{\pi/2} \int_0^2 2\underbrace{(r\sin(\theta))^2}_{r^2\sin^2(\theta)} r\, dr\, d\theta$$

$$= 2\left(\int_{\pi/4}^{\pi/2} \sin^2(\theta)\right)\left(\int_0^2 r^3\, dr\right)$$

$$= 2\int_{\pi/4}^{\pi/2} \frac{1}{2} - \frac{1}{2}\cos(2\theta)\, d\theta \quad \left[\frac{r^4}{4}\right]_0^2$$

Answer	$\pi + 2$

$$= 2\left[\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right]_{\pi/4}^{\pi/2} \left[\frac{16}{4}\right]$$

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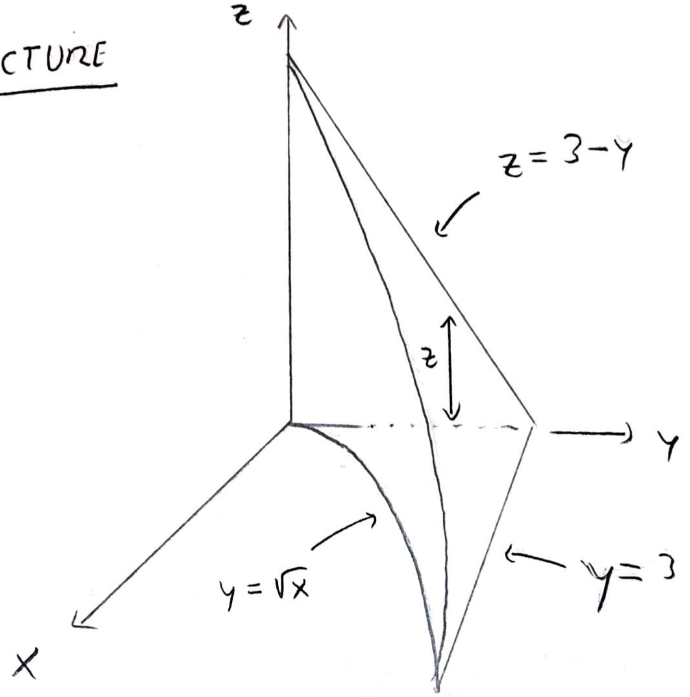
$$= 2\left(\frac{\pi}{4} - \frac{\pi}{8} - \frac{1}{4}\cancel{\sin(\pi)}^{0} + \frac{1}{4}\sin(\pi/2)\right)(4)$$

$$= 2\left(\pi/8 + \frac{1}{4}\right)/(4) = 8\left(\frac{\pi}{8} + \frac{1}{4}\right) = \pi + 2$$

3. *(10 points)* Set up, but do **NOT** evaluate, a triple integral that calculates the mass of an object E with density $2y$.

Here E is the region in the first octant bounded by the surfaces $y + z =$ [redacted] 3, $y = \sqrt{x}$, [redacted]. AND $y=3$ Write your integral in the usual order, with $dzdydx$.

1) PICTURE

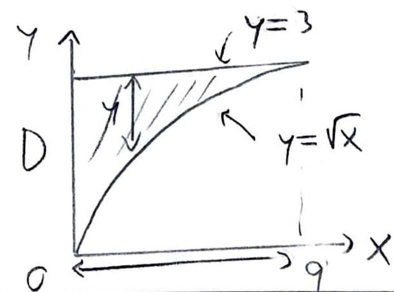


2) INEQUALITIES

SMALL ≤ z ≤ BIG

$0 \le z \le 3-y$

3) FIND D



$$\begin{cases} 0 \le z \le 3-y \\ \sqrt{x} \le y \le 3 \\ 0 \le x \le 9 \end{cases}$$

Answer	$\int_0^9 \int_{\sqrt{x}}^{3} \int_0^{3-y} 2y \, dz\, dy\, dx$

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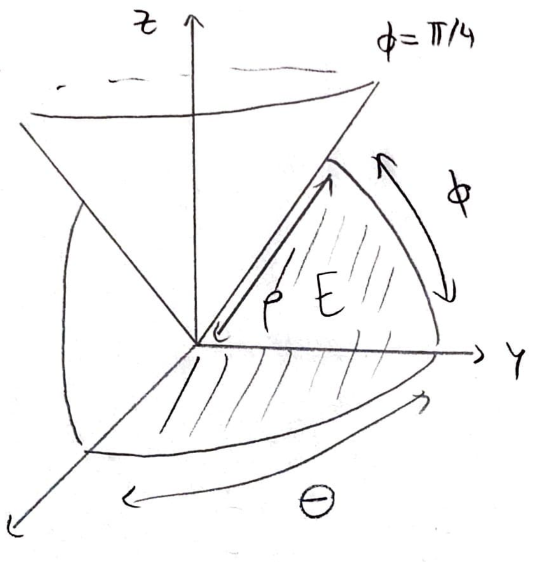
4. *(10 points)* Evaluate the following integral, where E is the region inside the sphere $x^2 + y^2 + z^2 =$ [redacted] 4 and below the cone $z = \sqrt{x^2 + y^2}$, in the first octant

$$\iiint_E 4z \, dxdydz$$

1) PICTURE



2) INEQUALITIES

$$\begin{cases} 0 \leq \rho \leq 2 \\ 0 \leq \theta \leq \pi/2 \\ \pi/4 \leq \phi \leq \pi/2 \end{cases}$$

3) $$\iiint_E 4z \, dxdydz = \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^2 4\rho \cos(\phi) \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi$$

$$= \frac{\pi}{2} \left(\int_0^2 4\rho^3 \, d\rho \right) \left(\int_{\pi/4}^{\pi/2} \cos(\phi) \sin(\phi) \, d\phi \right)$$

Answer	2π

$$= \frac{\pi}{2} \left[\rho^4\right]_0^2 \left[-\frac{1}{2} \cos^2(\phi)\right]_{\pi/4}^{\pi/2}$$ $\rightarrow \sin^2(\phi)/2$ ALSO OK

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$$= \left(\frac{\pi}{2}\right)(2^4)\left(-\frac{1}{2}(0) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2\right)$$

$$= \left(\frac{\pi}{2}\right)(2^4)\left(\frac{1}{4}\right) = \frac{16\pi}{8} = 2\pi$$

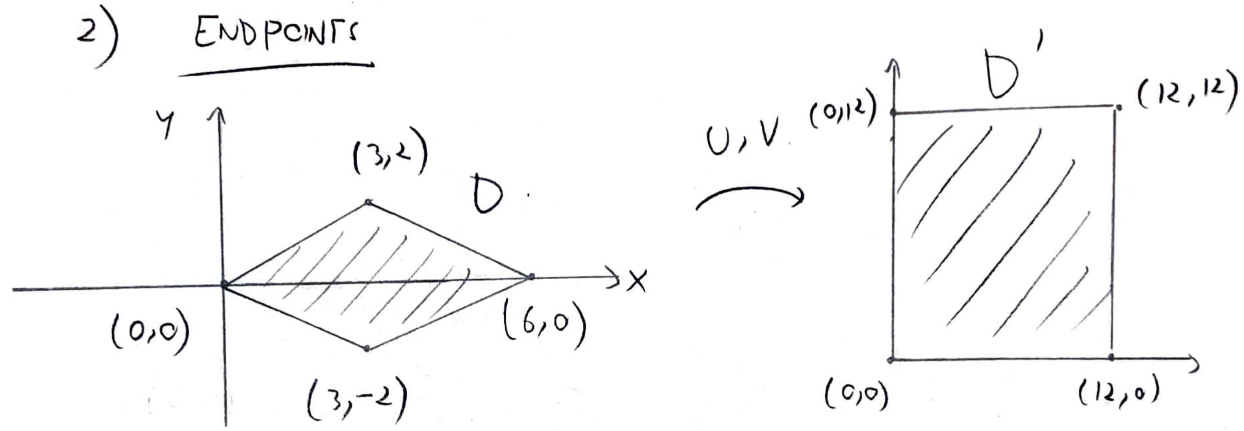
5. *(10 points)* Use a change of variables to evaluate the following double integral, where D is the ~~square~~ with vertices $(0,0)$, $(3,2)$, $(3,-2)$, $(6,0)$
DIAMOND

$$\iint_D (2x+3y)^3(2x-3y)^2\,dxdy$$

Simplify as much as possible, but leave your answer ~~in the form a^b or $-a^b$~~
AS A POWER

1) $\begin{cases} U = 2x + 3y \\ V = 2x - 3y \end{cases}$

2) ENDPOINTS



$(0,0) \rightsquigarrow U=0,\ V=0 \rightsquigarrow (0,0)$

$(3,2) \rightsquigarrow U=(2)(3)+(3)(2)=12,\quad V=2(3)-3(2)=0 \rightsquigarrow (12,0)$

$(3,-2) \rightsquigarrow U=(2)(3)+(3)(-2)=0,\quad V=2(3)-3(-2)=12 \rightsquigarrow (0,12)$

$(6,0) \rightsquigarrow U=(2)(6)+3(0)=12,\quad V=(2)(6)-3(0)=12 \rightsquigarrow (12,12)$

Answer	12^5

3) JACOBIAN

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$dUdV = \left|\dfrac{dUdV}{dxdy}\right| dxdy = |-12|\,dxdy = 12\,dxdy$

$\dfrac{dUdV}{dxdy} = \begin{vmatrix} \partial U/\partial x & \partial U/\partial y \\ \partial V/\partial x & \partial V/\partial y \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -6-6=-12$ (SEE BACK)

$$\Rightarrow dx\,dy = \frac{1}{12}\,du\,dv$$

4) INTEGRATE

$$\iint_D (2x+3y)^3 (2x-3y)^2 \, dx\,dy$$

$$= \iint_{D'} u^3 v^2 \left(\frac{1}{12}\right) du\,dv$$

$$= \frac{1}{12} \int_0^{12}\int_0^{12} u^3 v^2 \, du\,dv$$

$$= \frac{1}{12} \left(\int_0^{12} u^3 \, du\right)\left(\int_0^{12} v^2 \, dv\right)$$

$$= \frac{1}{12} \left[\frac{u^4}{4}\right]_0^{12} \left[\frac{v^3}{3}\right]_0^{12}$$

$$= \frac{1}{12} \; \frac{12^4}{4} \; \frac{12^3}{3}$$

$$= \frac{12^7}{(12)(12)}$$

$$= 12^5$$