MIDTERM REVIEW

1. Uniform Convergence

Problem 1: Let $f_n(x) = \left(x - \frac{1}{n}\right)^2$ on $[0,1]$. Show that f_n converges uniformly to some function f on $[0,1]$.

STEP 1: Find f

For all x we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(x - \frac{1}{n}\right)^2 \overset{\text{CONT}}{=} \left(\lim_{n\to\infty}\left(x - \frac{1}{n}\right)\right)^2 = x^2$$

Therefore $f(x) = x^2$

STEP 2: Uniform Convergence:

$$\begin{aligned}
|f_n(x) - f(x)| &= \left|\left(x - \frac{1}{n}\right)^2 - x^2\right| \\
&= \left|x^2 - \frac{2x}{n} + \frac{1}{n^2} - x^2\right| \\
&= \left|-\frac{2x}{n} + \frac{1}{n^2}\right| \\
&\leq \frac{2|x|}{n} + \frac{1}{n^2} \\
&\leq \frac{2}{n} + \frac{1}{n^2} \qquad \text{Since } 0 \leq x \leq 1
\end{aligned}$$

Date: Monday, July 25, 2022.

Let $\epsilon > 0$ be given, then since $\lim_{n\to\infty} \frac{2}{n} + \frac{1}{n^2} = 0$ there is N such that if $n > N$ then $\left|\frac{2}{n} + \frac{1}{n^2}\right| < \epsilon$.

With that N, if $n > N$ then by the above, for all x, we have

$$|f_n(x) - f(x)| \leq \frac{2}{n} + \frac{1}{n^2} < \epsilon$$

Hence f_n converges uniformly to f on $[0, 1]$ $\square$

2. Equicontinuity and Arzelà-Ascoli

Problem 2: Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of smooth functions such that (f_n') is bounded. Moreover, suppose for some x_0, $(f_n(x_0))$ is a bounded sequence. Show that (f_n) has a subsequence which converges uniformly on $[a, b]$.

We would like to apply Arzelá-Ascoli Theorem, so we need to show that (f_n) is bounded and equicontinuous.

STEP 1: Equicontinuous[1]

Let $M = \sup_x |f_n'(x)|$ and let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{M}$, then if $|x - y| < \delta$ then by the Mean-Value Theorem (c is between x and y)

$$|f_n(x) - f_n(y)| = |f_n'(c)|\,|x - y| \leq M\,|x - y| < M\left(\frac{\epsilon}{M}\right) = \epsilon \checkmark$$

STEP 2: Uniformly Bounded

Let $C = \sup_n |f_n(x_0)|$ then for all x, we have

[1] We have shown this in lecture, but we will need this idea in the next step

$$\begin{aligned}|f_n(x)| &= |f_n(x) - f_n(x_0) + f_n(x_0)| \\ &\leq |f_n(x) - f_n(x_0)| + |f_n(x_0)| \\ &\leq |f_n(c)|\,|x - x_0| + C \\ &\leq M(b-a) + C \checkmark\end{aligned}$$

STEP 3: Therefore, by the Arzelà-Ascoli Theorem, (f_n) has a subsequence which converges uniformly on $[a,b]$

3. Power Series

Problem 3: Show that $\sum_{n=1}^{\infty} \frac{x^n}{1+x^n}$ converges uniformly on $[0,a]$ for each $0 < a < 1$ but not uniformly on $[0,1)$

$$\left|\frac{x^n}{1+x^n}\right| = \frac{x^n}{1+x^n} \leq x^n \leq \underbrace{a^n}_{M_n}$$

But since $\sum_{n=1}^{\infty} a^n$ converges since $|a| < 1$ (Geometric series), by the Weierstraß M-test, the series converges.

If the series converged uniformly on $[0,1)$, it would be uniformly Cauchy, so with $\epsilon = \frac{1}{4}$ there is N such that if $n \geq m \geq N$ then for all $x \in [0,1)$

$$\sum_{k=m}^{n} \frac{x^k}{1+x^k} < \frac{1}{4}$$

In particular if you let $m = n = N$ then for all $x \in [0,1)$, we have

$$\frac{x^N}{1+x^N} < \frac{1}{4}$$

But this contradicts the fact that

$$\lim_{x \to 1} \frac{x^N}{1 + x^N} = \frac{1}{2}$$

4. Fourier Series

Problem 4: Suppose f is periodic of period 2π and continuously differentiable (so f' is continuous). Show that

$$\lim_{n \to \pm\infty} n\hat{f}(n) = 0$$

$$\begin{aligned}
\widehat{f'}(n) =& \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \\
=& \frac{1}{2\pi} \left[f(x) e^{-inx} \right]_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(-in) e^{-inx} dx \\
=& \frac{1}{2\pi} \left(f(\pi) e^{-in\pi} - f(-\pi) e^{in\pi} \right) + \frac{in}{2\pi} \int -\pi^{\pi} f(x) e^{-inx} dx \\
=& in\hat{f}(n)
\end{aligned}$$

(The terms in brackets cancel out by periodicity)

Therefore $\widehat{f'}(n) = in\hat{f}(n)$

But since f' is continuous on $[-\pi, \pi]$ it is bounded and integrable, and therefore by the Riemann-Lebesgue Lemma $\lim_{n \to \pm\infty} \widehat{f'}(n) = 0$ and so

$$\lim_{n \to \pm\infty} n\hat{f}(n) = \frac{1}{i} \lim_{n \to \pm\infty} in\hat{f}(n) = \frac{1}{i} \lim_{n \to \pm\infty} \widehat{f'}(n) = 0$$

5. Fourier Transform

Problem 5: Suppose $f \in \mathcal{S}$ and for all x we have

$$\int_{-\infty}^{\infty} f(y)e^{-y^2+2xy}dy = 0$$

Show that $f = 0$

Hint: Consider $f \star e^{-x^2}$

Using the hint, we calculate

$$\begin{aligned} f \star e^{-x^2} &= \int_{-\infty}^{\infty} f(y)e^{-(x-y)^2}dy \\ &= \int_{-\infty}^{\infty} f(y)e^{-x^2+2xy-y^2}dy \\ &= e^{-x^2}\int_{-\infty}^{\infty} f(y)e^{2xy-y^2}dy \\ &= 0 \end{aligned}$$

Hence $f \star e^{-x^2} = 0$ and hence

$$\begin{aligned} \widehat{f \star e^{-x^2}} &= \hat{0} \\ \widehat{f}\ \widehat{e^{-x^2}} &= 0 \\ \widehat{f}\ e^{-x^2} &= 0 \\ \widehat{f} &= 0 \end{aligned}$$

And therefore $f = 0$[2]

[2] This follows from the Fourier inversion formula $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$