SOLUTIONS

VERSION A

1. $\displaystyle\int_0^{2\pi}\int_0^{2}\int_{-\sqrt{r^2+5}}^{\sqrt{r^2+5}} zr\,dz\,dr\,d\theta$

2. $\frac{15}{4}\left(\frac{\pi}{12}-\frac{\sqrt{3}}{8}\right)$

3. (a) NO

   (b) 1

4. 0

5. $\frac{7}{3}\left(\cosh(2)-1\right)$

VERSION B

1. $\displaystyle\int_0^{2\pi}\int_0^{3}\int_{-\sqrt{r^2+16}}^{\sqrt{r^2+16}} zr\,dz\,dr\,d\theta$

2. $\frac{15}{4}\left(\frac{\pi}{12}-\frac{\sqrt{3}}{8}\right)$

3. (a) NO

   (b) 1

4. 0

5. $\frac{26}{3}\left(\cosh(3)-1\right)$

MATH 2E – MIDTERM

Name: ______________________

Student ID: ______________________

Discussion Section time: (please circle)

11 – 12 PM 2 – 3 PM

Instructions: Welcome to your Midterm! You have 50 minutes to take this exam, for a total of 100 points. No books, notes, calculators, or cellphones are allowed. Remember that you are not only graded on your final answer, but also on your work. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. May your luck be conservative! :)

Academic Honesty Statement: I hereby certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating (no matter how small) results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to section 102.1 of the UCI Student Code of Conduct.

Signature: ______________________________

1		20
2		20
3(a)		10
3(b)		10
4		20
5		20
Total		100

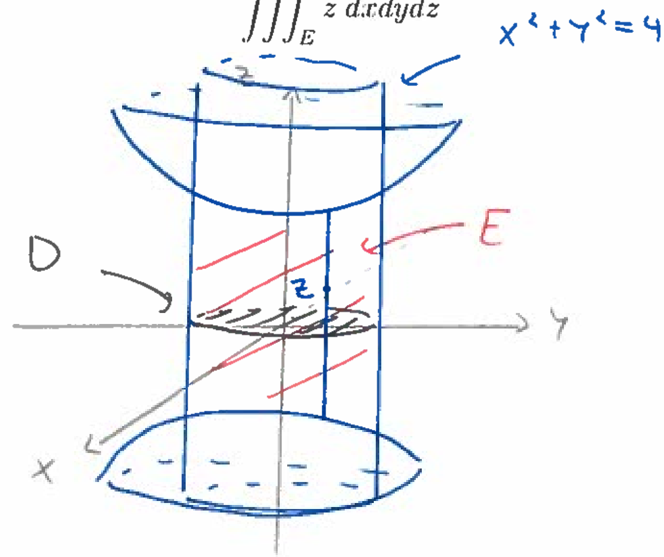
Date: Friday, November 2, 2018.

CYLINDER HYPERBOLOID OF 2 SHEETS (2 CUPS)

1. *(20 points)* Set up, but do **NOT** evaluate, the integral below, where E is the solid bounded by the surfaces $x^2+y^2=4$ and $z^2-x^2-y^2=5$. Include a (rough) picture of E, and simplify the endpoints of integration as much as possible.

$$\iiint_E z\, dxdydz$$

1) PICTURE



2) SMALL $\leq z \leq$ BIG

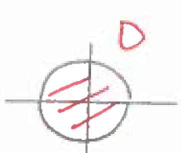
$z^2-x^2-y^2=5 \Rightarrow z = \pm\sqrt{x^2+y^2+5} = \pm\sqrt{r^2+5}$

$-\sqrt{r^2+5} \leq z \leq \sqrt{r^2+5}$

3) FIND D D: DISK OF RADIUS 2 (SEE PICTURE, OR INTERSECT $x^2+y^2=4$ AND $z^2-x^2-y^2=5$)

$0 \leq r \leq 2$

$0 \leq \theta \leq 2\pi$

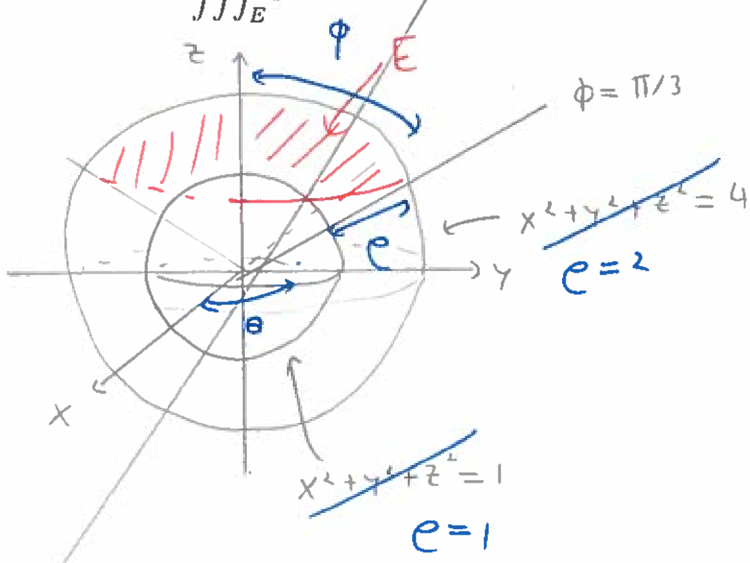


4) ANS = $\boxed{\int_0^{2\pi}\int_0^{2}\int_{-\sqrt{r^2+5}}^{\sqrt{r^2+5}} z\, r\, dz\, dr\, d\theta}$

2. *(20 points)* Calculate the following integral, where E is the solid in the first octant above the surface $\phi = \frac{\pi}{3}$ and between the surfaces $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Include a (rough) picture.

$$\iiint_E y\,dxdydz$$

1) PICTURE



2) $1 \le \rho \le 2$

$0 \le \theta \le \pi/2$

$0 \le \phi \le \pi/3$

3) ANS

$$\int_0^{\pi/3}\int_0^{\pi/2}\int_1^2 \overbrace{\rho\sin(\phi)\sin(\theta)}^{y}\,\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

$$= \left(\int_1^2 \rho^3\,d\rho\right)\left(\int_0^{\pi/2}\sin(\theta)\,d\theta\right)\left(\int_0^{\pi/3}\underbrace{\sin^2(\phi)}_{\frac{1}{2}-\frac{1}{2}\cos(2\phi)}\,d\phi\right)$$

$$= \left[\frac{\rho^4}{4}\right]_1^2\left[-\cos(\theta)\right]_0^{\pi/2}\left[\frac{\phi}{2}-\frac{\sin(2\phi)}{4}\right]_0^{\pi/3}$$

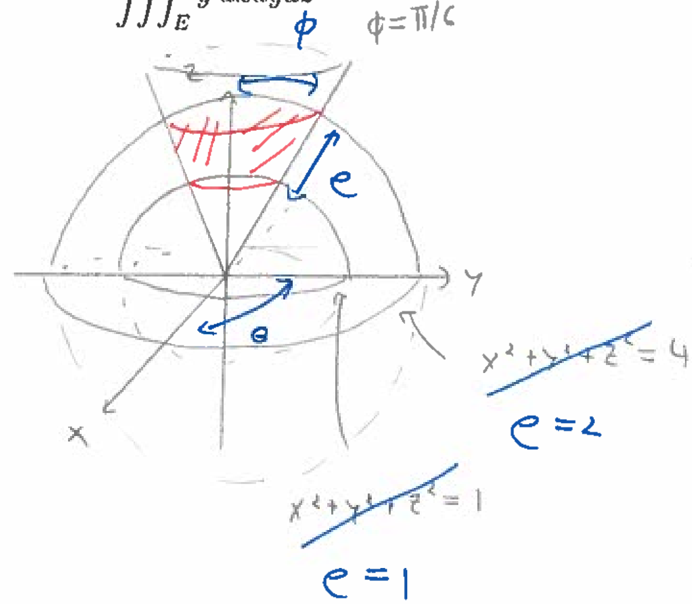
$$= \left(\frac{15}{4}\right)(1)\left(\frac{\pi}{6} -\right.$$

$$= \frac{15\pi}{24} - \frac{15\pi}{}$$

2. *(20 points)* Calculate the following integral, where E is the solid in the first octant above the surface $\phi = \frac{\pi}{6}$ and between the surfaces $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Include a (rough) picture.

$$\iiint_E y \, dxdydz$$

1) PICTURE



2) $1 \leq \rho \leq 2$

$0 \leq \theta \leq \pi/2$

$0 \leq \phi \leq \pi/6$

3) $$\iiint_E y \, dx\,dy\,dz = \int_0^{\pi/6}\int_0^{\pi/2}\int_1^2 \underbrace{\rho \sin(\phi)\sin(\theta)}_{y} \, \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi$$

$$= \left(\int_1^2 \rho^3 d\rho\right)\left(\int_0^{\pi/2} \sin(\theta)\, d\theta\right)\left(\int_0^{\pi/6} \sin^2(\phi)\, d\phi\right)$$

$\sin^2(\phi) = \frac{1}{2} - \frac{\cos(2\phi)}{2}$

$$= \left[\frac{1}{4}\rho^4\right]_1^2 \left[-\cos(\theta)\right]_0^{\pi/2} \left[\frac{\phi}{2} - \frac{\sin(2\phi)}{4}\right]_0^{\pi/6}$$

$$= \frac{1}{4}(16-1)\,(0+1)\left(\frac{\pi/6}{2} - \frac{\sin(\pi/3)}{4}\right)$$

ACCEPTABLE ANSWER

$$= \boxed{\left(\frac{15}{4}\right)\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right)} = \frac{15\pi}{48} - \frac{15\sqrt{3}}{32} = \frac{5\pi}{16} - \frac{15\sqrt{3}}{32}$$

3(a) *(10 points)* Is the following vector field conservative? Why or why not?

$$\mathbf{F}(x,y) = \left\langle \frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

$$\frac{\partial P}{\partial y} = \frac{\partial\left(\frac{-x}{x^2+y^2}\right)}{\partial y} = \frac{2xy}{(x^2+y^2)^2}$$

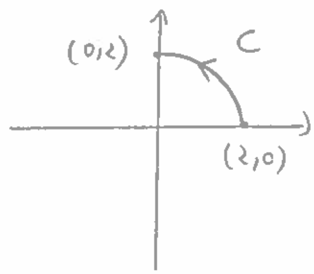
$$\frac{\partial Q}{\partial x} = \frac{\partial\left(\frac{-y}{x^2+y^2}\right)}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$, so NO

3(b) *(10 points)* Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the part of the circle centered at $(0, 0)$ and radius 2 in the first quadrant, oriented counterclockwise, and

$$\mathbf{F}(x, y) = \left\langle \frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

1) PICTURE



2) $x(t) = 2\cos(t)$ $\quad 0 \leq t \leq \pi/2$

$y(t) = 2\sin(t)$

3) $$\int_C F \cdot dr = \int_0^{\pi/2} F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^{\pi/2} \left\langle \frac{-x(t)}{(x(t))^2+(y(t))^2}, \frac{y(t)}{(x(t))^2+(y(t))^2} \right\rangle \cdot \langle x'(t), y'(t) \rangle d$$

$$= \int_0^{\pi/2} \left\langle -\frac{2\cos(t)}{4}, \frac{2\sin(t)}{4} \right\rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle dt$$

$$= \int_0^{\pi/2} \frac{4}{4}\cos(t)\sin(t) + \frac{4}{4}\sin(t)\cos(t)\, dt$$

$$= \int_0^{\pi/2} 2\cos(t)\sin(t)\, dt$$

$$= \int_0^{\pi/2} \sin(2t)\, dt = \left[-\frac{\cos(2t)}{2} \right]_0^{\pi/2}$$

$= 1/2 + 1/2 = 1$

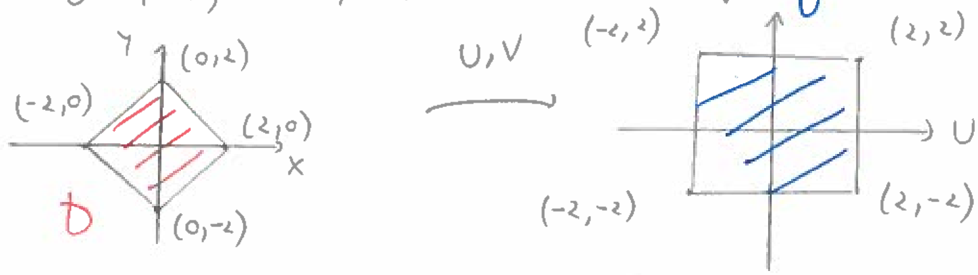
4. *(20 points)* Calculate the following integral, where D is the square with vertices $(2,0), (0,2), (-2,0), (0,-2)$.

$$\iint_D \sin\left(y^2 - x^2\right) dx dy$$

Hint: $y^2 - x^2 = (y-x)(y+x)$

1) $U = Y - X, \quad V = Y + X$

2)



$(2,0) \rightsquigarrow U = 0-2 = -2, \quad V = 0+2 = 2 \rightsquigarrow (-2,2)$

$(0,2) \rightsquigarrow (2,2)$

$(-2,0) \rightsquigarrow (2,-2)$

$(0,-2) \rightsquigarrow (-2,-2)$

3) $dUdV = \left|\dfrac{dUdV}{dXdY}\right| dXdY = |-2|\, dXdY = 2\, dXdY \Rightarrow dXdY = \frac{1}{2} dUdV$

$$\frac{dUdV}{dXdY} = \begin{vmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

3) $\displaystyle\iint_D \sin(Y^2 - X^2)\, dXdY = \iint_{D'} \sin(UV) \left(\frac{1}{2} dUdV\right)$

$$= \frac{1}{2} \int_{-2}^{2}\int_{-2}^{2} \sin(UV)\, dUdV$$

$$= \frac{1}{2}\int_{-2}^{2} \left[-\frac{\cos(UV)}{V}\right]_{U=-2}^{U=2} dV$$

$\boxed{0}$

$\|$

$$= \frac{1}{2}\int^{2} -\frac{\cos(2V)}{V} + \frac{\cos(-2V)}{V}\, dV = \frac{1}{2}\int^{2} -\cancel{\frac{\cos(2V)}{V}} + \cancel{\frac{\cos(2V)}{V}}\, dV$$

5. *(20 points)* Let D be the region in the first quadrant of the xy–plane enclosed by the hyperbolas $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$ and the lines $y = 0$ and $y = \tanh(2)x$ (you may assume that $y = \tanh(2)x$ intersects each hyperbola in exactly one point). Using the hyperbolic coordinates below, evaluate the following integral. Include a (rough) picture of D. Remember to justify all your steps.

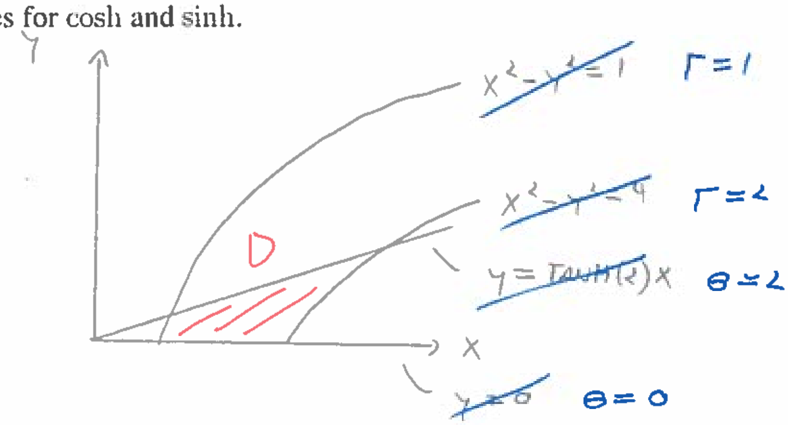
[redacted] $\iint_D y\,dx\,dy$

Hyperbolic coordinates: $(r \geq 0)$

$$x = r\cosh(\theta)$$
$$y = r\sinh(\theta)$$

Reminder: On the second page of the exam there are some useful identities for cosh and sinh.

1) PICTURE



2) $x^2 - y^2 = 1 \Rightarrow r^2\cosh^2(\theta) - r^2\sinh^2(\theta) = 1$

$\Rightarrow r^2(\underbrace{\cosh^2(\theta) - \sinh^2(\theta)}_{1}) = 1$

$\Rightarrow r^2 = 1 \Rightarrow \underline{r = 1}$

SIMILARLY, $x^2 - y^2 = 4 \Rightarrow \underline{r = 2}$

$y = 0 \Rightarrow \sinh(\theta) = 0 \Rightarrow \frac{e^{\theta} - e^{-\theta}}{2} = 0 \Rightarrow e^{\theta} = e^{-\theta} \Rightarrow \theta = -\theta \Rightarrow \underline{\theta = 0}$

$y = \tanh(2)x \Rightarrow \cancel{r}\sinh(\theta) = \tanh(2)\cancel{r}\cosh(\theta) \Rightarrow \frac{\sinh(\theta)}{\cosh(\theta)} = \tanh(2)$

$\Rightarrow \tanh(\theta) = \tanh(2)$

$\Rightarrow \underline{\theta = 2}$ (SINCE TANH IS $\underline{1\text{-}1}$)

SO $D: \ 1 \leq r \leq 2, \ 0 \leq \theta \leq 2$

3) $dx\,dy = \left|\frac{dx\,dy}{dr\,d\theta}\right| dr\,d\theta$

$$\frac{dx\,dy}{dr\,d\theta} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cosh(\theta) & r\sinh(\theta) \\ \sinh(\theta) & r\cosh(\theta) \end{vmatrix}$$

$$= r\cosh^2(\theta) - r\sinh^2(\theta)$$

$$= r\left(\cosh^2(\theta) - \sinh^2(\theta)\right)$$

$$= r$$

so $dx\,dy = |r|\,dr\,d\theta = r\,dr\,d\theta$

4) <u>ANS</u> $$\int_0^2 \int_1^2 r\sinh(\theta)\, r\, dr\, d\theta$$

$$= \left(\int_1^2 r^2\,dr\right)\left(\int_0^2 \sinh(\theta)\,d\theta\right)$$

$$= \left[\frac{r^3}{3}\right]_1^2 \left[\cosh(\theta)\right]_0^2$$

$$= \left(\frac{7}{3}\right)\left(\cosh(2) - \cosh(0)\right)$$

$$\cosh(0) = \frac{e^0 + e^{-0}}{2} = 1$$

$$= \boxed{\left(\frac{7}{3}\right)\left(\cosh(2) - 1\right)}$$