MATH S4062 – MIDTERM – SOLUTIONS

1. **Banach Fixed Point Theorem:** If (X, d) is complete and f is a contraction, then f has a unique fixed point p.

   STEP 1: Let $x_0 \in X$ and define $x_n = f^n(x_0)$

   Notice $d(x_1, x_2) = d(f(x_0), f(x_1)) \leq kd(x_0, x_1)$ and

   $d(x_2, x_3) = d(f(x_1), f(x_2)) \leq kd(x_1, x_2) \leq kkd(x_0, x_1) = k^2 d(x_0, x_1)$

   More generally $d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$

   STEP 2: Claim: (x_n) is Cauchy

   Why? Let $\epsilon > 0$ be given and N be TBA, then if $m, n > N$ (WLOG assume $n \geq m$), then

$$
\begin{aligned}
d(x_m, x_n) \leq & d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\
\leq & k^m d(x_0, x_1) + k^{m+1} d(x_0, x_1) + \cdots + k^{n-1} d(x_1, x_0) \qquad \text{(By STEP 1)} \\
\leq & \left(k^m + k^{m+1} + \cdots + k^{n-1}\right) d(x_1, x_0) \\
= & k^m \left(1 + k + \cdots + k^{n-m-1}\right) d(x_0, x_1) \\
\leq & k^m \left(1 + k + k^2 + \cdots\right) d(x_0, x_1) \\
= & k^m \left(\frac{1}{1-k}\right) d(x_0, x_1) \\
\leq & \frac{k^N}{1-k} d(x_0, x_1) \qquad \text{Since } m > N \text{ and } k < 1
\end{aligned}
$$

   But since $k < 1$ we have $\lim_{n\to\infty} k^n = 0$, so we can choose N large enough so that $\frac{k^N}{1-k} d(x_0, x_1) < \epsilon$, which in turn implies $d(x_m, x_n) < \epsilon$ ✓

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STEP 3: Since (x_n) is Cauchy and X is complete, (x_n) converges to some p

Claim: p is a fixed point of f.

This follows because

$$\begin{aligned} x_{n+1} &= f(x_n) \\ \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} f(x_n) \\ p &= f\left(\lim_{n\to\infty} x_n\right) \qquad \text{(continuity)} \\ p &= f(p) \checkmark \end{aligned}$$

STEP 4: **Uniqueness:** Suppose there are two fixed points $p \neq q$, then

$$d(p,q) = d(f(p), f(q)) \leq k d(p,q) < d(p,q)$$

Which is a contradiction ☐

2. Suppose there are $x \neq y$ with $f(x) \neq f(y)$ and let $\epsilon = |f(x) - f(y)|$

Since f_n is equicontinuous on $[-1,1]$ there is $\delta > 0$ such that if $|x_0 - y_0| < \delta$ then for all n we have $|f_n(x_0) - f_n(y_0)| < \epsilon$.

However, with x and y as above, then since $\lim_{n\to\infty} \frac{x}{n} - \frac{y}{n} = 0$ there is n large enough so that $\left|\frac{x}{n} - \frac{y}{n}\right| < \delta$, and both are in $[-1,1]$, and therefore

$$\begin{aligned}
\left|f_n\left(\frac{x}{n}\right) - f_n\left(\frac{y}{n}\right)\right| &< \epsilon \\
\left|f\left(n\left(\frac{x}{n}\right)\right) - f\left(n\left(\frac{y}{n}\right)\right)\right| &< \epsilon \\
|f(x) - f(y)| &< \epsilon \\
|f(x) - f(y)| &< |f(x) - f(y)|
\end{aligned}$$

Which is a contradiction, and therefore f is constant.

3. Notice that if $x \geq a$ then $1+\left(n^2\right)x \geq \left(n^2\right)x \geq n^2 a$ and therefore

$$\frac{1}{1+(n^2)\,x} \leq \frac{1}{n^2 a} = \underbrace{\frac{1}{a}\left(\frac{1}{n^2}\right)}_{M_n}$$

Therefore, since

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{a\,(n^2)} = \frac{1}{a}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Converges (it's a $2-$series), by the Weierstraß M-test, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{1+(n^2)\,x} \text{ converges uniformly on } [a,\infty)$$

However, it does not converge uniformly on $(0,\infty)$ because if it did, then it would be uniformly Cauchy, and so by the Cauchy criterion for series, there would be N such that if with $\epsilon = \frac{1}{2}$ there is N such that if $n \geq m \geq N$ then for all $x > 0$

$$\sum_{k=m}^{n} \frac{1}{1+(n^2)\,x} < \frac{1}{2}$$

In particular if you let $m = n = N$ then for all $x > 0$, we have

$$\frac{1}{1+(N^2)\,x} < \frac{1}{2}$$

But this contradicts the fact that

$$\lim_{x\to 0} \frac{1}{1+(N^2)\,x} = 1$$

4. Let $\epsilon > 0$ be given, then since $f_n \to f$ uniformly, there is N_1 such that if $n > N_1$ then for all y we have $|f_n(y) - f(y)| < \frac{\epsilon}{4M}$

   Moreover, since $\lim_{n\to\infty} \int_{|y|>M} |f_n(y) - f(y)|\, dy = 0$ there is N_2 such that if $n > N_2$ then $\int_{|y|>M} |f_n(y) - f(y)|\, dy < \frac{\epsilon}{2}$

   Let $N = \max\{N_1, N_2\}$ then if $n > N$ then for all x we have

$$\begin{aligned}
\left|\widehat{f_n}(x) - \widehat{f}(x)\right| &= \left|\int_{-\infty}^{\infty} f_n(y)e^{-2\pi ixy}dy - f(y)e^{-2\pi ixy}dy\right| \\
&= \left|\int_{-\infty}^{\infty} \left(f_n(y) - f(y)\right) e^{-2\pi ixy}dy\right| \\
&\leq \int_{-\infty}^{\infty} |f_n(y) - f(y)| \underbrace{\left|e^{-2\pi ixy}\right|}_{=1} dy \\
&= \int_{|y|\leq M} \underbrace{|f_n(y) - f(y)|}_{<\frac{\epsilon}{4M}} dy + \underbrace{\int_{|y|>M} |f_n(y) - f(y)|\, dy}_{<\frac{\epsilon}{2}} \\
&< \frac{\epsilon}{4M} \int_{|y|\leq M} 1 dy + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{4M}(2M) + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon \checkmark
\end{aligned}$$

   Therefore $\widehat{f_n} \to \widehat{f}$ uniformly