MATH S4062 – MIDTERM STUDY GUIDE

The midterm will take place on **Tuesday, July 26, 2022** from 10:45 am to 12:20 pm in our usual lecture room. It is a closed book and closed notes exam, and counts for 25% of your grade.

This exam covers everything up to and **including** Lecture 10. This corresponds to Chapters 7 and 8 of Rudin, as well as the material about Fourier series and the Fourier transform, and the material from Homework $1 - 5$

There will be 4 questions in total (some with multiple parts), so expect to spend roughly 20-25 mins per question.

- The first one will be one of the "Proofs you should know" below.
- The second question will be taken directly from the homework.
- The last two questions will (probably) be new questions.

Within a problem, I might ask you to restate some of the definitions below, or provide examples or counterexamples

Date: Tuesday, July 26, 2022.

Proofs you should know

Know how to state and prove the following theorems. I could theory ask you to reprove any of the below (or variations thereof):

(1) If $f_n \to f$ uniformly and each f_n is continuous at x_0 then f is continuous at x_0

(2) Weierstraß $M-$Test

(3) The Banach Fixed Point Theorem

(4) If $r < R$, the power series of f converges uniformly on $[-r, r]$

(5) $\widehat{f \star g}(n) = \hat{f}(n)\hat{g}(n)$

(6) The Orthogonal Decomposition Theorem and its corollaries: the Bessel's Inequality, the Riemann-Lebesgue Lemma, and Parseval's Theorem. (only the proofs, no need to memorize the statements, and you do not need to reprove the facts used in those theorems)

Definitions you should know

Know how to define the following terms. You do not need to know the proofs of the theorems

(1) $f_n \to f$ pointwise

(2) $f_n \to f$ uniformly

(3) A sequence of functions that converges pointwise but not uniformly ($f_n = x^n$ on $[0, 1]$ works)

(4) The $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ example, whose uniform limit is not differentiable

(5) $C[a, b]$ and its metric d

(6) $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly

(7) (f_n) is bounded

(8) (f_n) is equicontinuous

(9) An example of a non-equicontinuous sequence. $\sin(nx)$ on $[-1, 1]$would work, by a result from the homework

(10) Arzelà-Ascoli Theorem

(11) Fixed points, Contraction

(12) An example of a function with $|f'(x)| < 1$ for all x but with no fixed point (see Homework 2)

(13) Weierstraß Approximation Theorem

(14) Power series, Radius of Convergence

(15) Fubini for series, including the counterexample

(16) Fourier series on $[-\pi, \pi]$, $\hat{f}(n)$ on $[-\pi, \pi]$, $S_N(f)$

(17) $f \star g$, both on $[-\pi, \pi]$ and on $(-\infty, \infty)$

(18) Self-Adjointness of the Fourier transform

Trig Formulas you should know

Here is a list of trig identities you should know. Anything else will be provided

(1) $\cos^2(x) + \sin^2(x) = 1$

(2) $1 + \tan^2(x) = \sec^2(x)$

(3) Derivatives of $\sin, \cos, \tan, \sec$

(4) Antiderivatives of $\sin, \cos, \tan$

(5) $\cos(-x) = \cos(x), \sin(-x) = -\sin(x)$, $\tan(-x) = -\tan(x)$

(6) $\cos(2x) = \cos^2(x) - \sin^2(x)$

(7) $\sin(2x) = 2\sin(x)\cos(x)$

(8) $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

(9) $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

(10) $\int \cos^2(x)dx, \int \sin^2(x)dx$

List of formulas that will be provided

Here is a list of formulas that I will provide if necessary. You do not need to memorize them.

(1) Fourier series and coefficients for intervals *other* than $[-\pi, \pi]$

(2) D_N and its explicit formula

(3) The Fourier Transform and all its properties, but I will not give you Self-Adjointness or the properties of $f \star g$

(4) The definition of a good kernel

(5) The definition of the Schwarz space

(6) All the formulas for $\cos(A)\cos(B)$, $\sin(A)\sin(B)$, $\cos(A)\sin(B)$

Things you can ignore

- Lecture 1: The proof in the section "Uniform Convergence and Integration"
- Lecture 1: The $nx\left(1-x^2\right)^n$ example **but** I could give you the formula and ask you to show $f_n \to 0$ pointwise on $[0,1]$ but $\int_0 f_n \nrightarrow 0$
- Lecture 2: The proof in the section "Uniform Convergence and Differentiation"
- Lecture 2: The $f_n = \frac{x^2}{(1+x^2)^n}$ example **but** I could ask you to show what $\sum f_n$ converges to.
- Lecture 3: The "Counterexample" section showing that $\sin(nx)$ has no convergent subsequence
- Lecture 3: The proof of the Arzelà-Ascoli Theorem
- Lecture 4: The proof of the Picard-Lindelöf Theorem, but notice how we wrote the ODE as a fixed point problem
- Lecture 4: For the continuous but nowhere differentiable function, I just want you to understand how the Weierstraß M test is used, the rest you can safely ignore

- Lecture 5: The proof of the Weierstraß Approximation Theorem
- Lecture 5: The terminology of the Stone-Weierstraß Theorem, I would give it to you if necessary
- Lecture 5: The proof of term-by-term integration and differentiation of series
- Lecture 6: The proof of Fubini for series
- Lecture 6: The statement and proof of Taylor's theorem
- Lecture 6: The section on the Gamma Function and the section on the Half-Derivative
- Lecture 7: The proof of uniqueness of Fourier series
- Lecture 8: the proof of the formula for D_N
- Lecture 8: The proof of pointwise convergence (the one with Lipschitz at x)
- Lecture 9: The proof of the Best Approximation Lemma and the proof of Mean-Squared convergence
- Lecture 10: The proofs in the section "Derivatives and Fourier transforms"
- Lecture 10: The proof of the Fourier Inversion Formula and of Plancherel's Theorem
- Homework 4: Ignore Additional Problems 1 and 2
- Homework 5: Ignore Additional Problems 2, 3, and 5