SOLUTIONS

1. *(10 points)* Using the change of variables below, find the area of the region D bounded by the ellipse $x^2 - xy + y^2 = 8$

$$\begin{cases} x = \sqrt{2}u - \sqrt{\frac{2}{3}}v \\ y = \sqrt{2}u + \sqrt{\frac{2}{3}}v \end{cases}$$

1) $\text{AREA}(D) = \iint_D 1 \, dx\,dy$

2) <u>FIND D'</u>

$x^2 - xy + y^2 = 8 \implies \left(\sqrt{2}U - \sqrt{\frac{2}{3}}V\right)^2 - \left(\sqrt{2}U - \sqrt{\frac{2}{3}}V\right)\left(\sqrt{2}U + \sqrt{\frac{2}{3}}V\right) + \left(\sqrt{2}U + \sqrt{\frac{2}{3}}V\right)^2 = 8$

$\implies \cancel{2U^2} - \cancel{2\frac{2}{\sqrt{3}}UV} + \frac{2}{3}V^2 - \cancel{2U^2} + \frac{2}{3}V^2 + 2U^2 + \cancel{2\frac{2}{\sqrt{3}}UV} + \frac{2}{3}V^2 = 8$

$\implies 2U^2 + \left(\frac{2}{3} + \frac{2}{3} + \frac{2}{3}\right)V^2 = 8$

$\implies 2U^2 + 2V^2 = 8$

$\implies \underline{U^2 + V^2 = 4}$ $\rightsquigarrow$ SO D' IS A DISK OF RADIUS 2

3) $dx\,dy = \left|\frac{dx\,dy}{dU\,dV}\right| dU\,dV$

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$$\frac{dx\,dy}{dU\,dV} = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{vmatrix} = \begin{vmatrix} \sqrt{2} & -\sqrt{\frac{2}{3}} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{vmatrix}$$

$$= \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

so $dx\,dy = \left|\frac{4}{\sqrt{3}}\right| dU\,dV = \frac{4}{\sqrt{3}}\,dU\,dV$

4) $\text{AREA}(D) = \iint_D 1\,dx\,dy$

$$= \iint_{D'} 1 \left(\frac{4}{\sqrt{3}}\right) dU\,dV$$

$$= \frac{4}{\sqrt{3}} \iint_{D'} 1\,dU\,dV$$

$$= \frac{4}{\sqrt{3}}\,\text{AREA}(D')$$

$D' =$ DISK OF RADIUS 2

$$= \frac{4}{\sqrt{3}}\,\pi(2)^2$$

$$= \frac{16\pi}{\sqrt{3}}$$

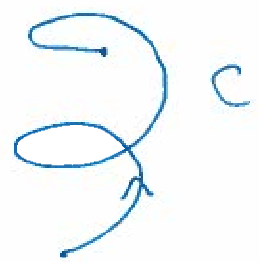
2. (*10 points*) Find $\int_C F \cdot dr$, where

$$F(x, y, z) = \langle \sin(y), (x\cos(y) + \cos(z)), -y\sin(z) \rangle$$

C is the helix with parametric equations (include a picture of C)

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \\ z(t) = t \\ 0 \le t \le 4\pi \end{cases}$$

1) PICTURE



2) F CONSERVATIVE ✓ (SEE SECTION 16.5)

3) ANTIDERIVATIVE $F = \nabla f = \langle f_x, f_y, f_z \rangle$

$= \langle \sin(y), x\cos(y) + \cos(z), -y\sin(z) \rangle$

$f_x = \sin(y) \Rightarrow f = \int \sin(y)\,dx = x\sin(y) + \text{JUNK}$

$f_y = x\cos(y) + \cos(z) \Rightarrow f = \int x\cos(y) + \cos(z)\,dy$

$= x\sin(y) + \cos(z)\,y + \text{JUNK}$

$f_z = -y\sin(z) \Rightarrow f = \int -y\sin(z)\,dz = y\cos(z) + \text{JUNK}$

HENCE $f(x,y,z) = x\sin(y) + y\cos(z)$

4) $\int_C F \cdot dr = \int_C \nabla f \cdot dr \overset{\text{FTC}}{=} f(x(4\pi), y(4\pi), z(4\pi)) - f(x(0), y(0), z(0))$

$= f(1, 0, 4\pi) - f(1, 0, 0)$

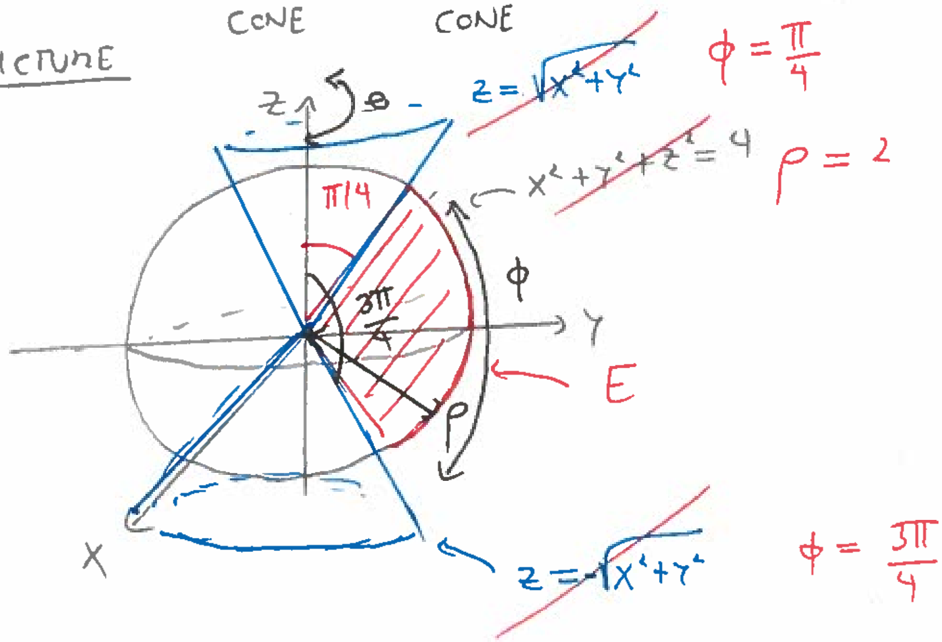
$= 1\sin(0) + 0\cos(4\pi) - 1\sin(0) - 0\cos(0) = 0$

3 · ~~2~~ *(10 points)* Calculate

$$\iiint_E y\,dxdydz$$

E is the solid in the region $y \geq 0$, inside $x^2+y^2+z^2=4$, between $z = \underbrace{-\sqrt{x^2+y^2}}_{\text{CONE}}$ and $z = \underbrace{\sqrt{x^2+y^2}}_{\text{CONE}}$. Include a picture of E.

1) PICTURE



⚠ $z = -\sqrt{x^2+y^2}$ CORRESPONDS TO $\phi = \frac{3\pi}{4}$, NOT $-\frac{\pi}{4}$,

SINCE $0 \leq \phi \leq \pi$

2) INEQUALITIES

$$0 \leq \rho \leq 2$$
$$0 \leq \theta \leq \pi$$
$$\frac{\pi}{4} \leq \phi \leq \frac{3\pi}{4}$$

3) $$\iiint_E y\,dxdydz = \int_{\pi/4}^{3\pi/4}\int_0^{\pi}\int_0^2 \rho \sin(\phi)\sin(\theta)\,\rho^2 \sin(\phi)\,d\rho\, d\theta\, d\phi$$

$$= \left(\int_{\pi/4}^{3\pi/4} \sin^2(\phi)\,d\phi\right)\left(\int_0^{\pi} \sin(\theta)\,d\theta\right)\left(\int_0^2 \rho^3\,d\rho\right)$$

$$= \left(\int_{\pi/4}^{3\pi/4} \frac{1}{2} - \frac{1}{2} \cos(2\phi)\, d\phi \right) \left[-\cos(\theta) \right]_0^{\pi} \left[\frac{\rho^4}{4} \right]_0^2$$

$$= \left[\frac{\phi}{2} - \frac{1}{4} \sin(2\phi) \right]_{\pi/4}^{3\pi/4} \underbrace{(1+1) \left(\frac{16}{4} \right)}_{8}$$

$$= \left(\frac{3\pi}{8} - \frac{1}{4} \sin\left(\frac{3\pi}{2}\right) - \frac{\pi}{8} + \frac{1}{4} \sin\left(\frac{\pi}{2}\right) \right) 8$$

$$= \left(\frac{2\pi}{8} + \frac{1}{4} + \frac{1}{4} \right) 8$$

$$= \left(\frac{\pi}{4} + \frac{1}{2} \right) 8$$

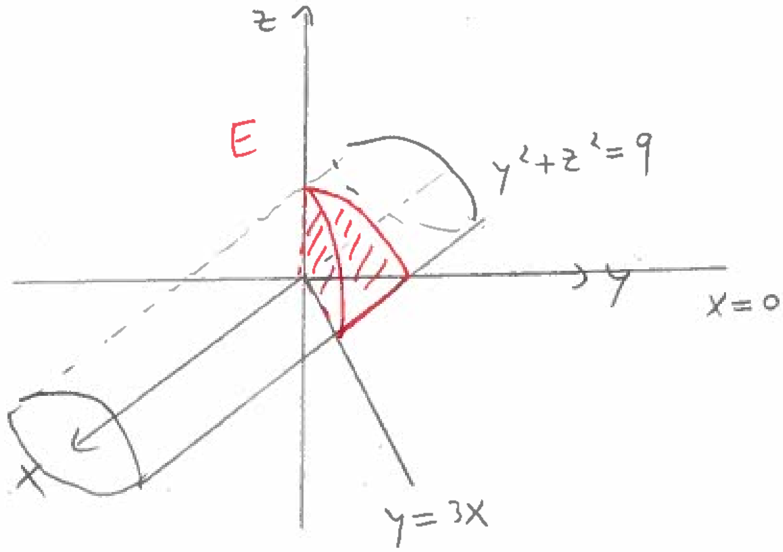
$$= \boxed{2\pi + 4}$$

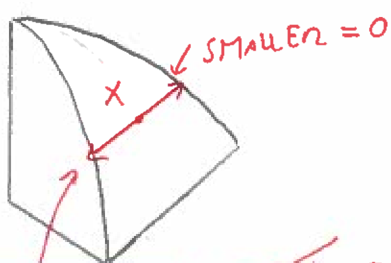
4. ~~3~~. *(10 points)* Calculate

$$\iiint_E 3\, dx\,dy\,dz$$

E is the solid bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0, y = 3x, z = 0$, in the first octant. Include a picture of E.

1) PICTURE

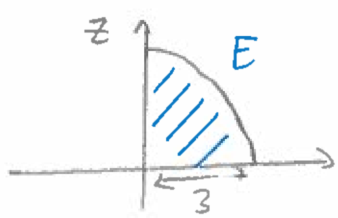




BIGGER : ~~$y=3x$~~ $\Longrightarrow x = \frac{y}{3}$

2) SMALLER $\leq x \leq$ BIGGER $\Longrightarrow$ $\boxed{0 \leq x \leq \frac{y}{3}}$

3) FIND D : D = SHADOW BEHIND E = QUARTER CIRCLE (USE $y^2+z^2=4$)



$\boxed{0 \leq r \leq 3 \quad , \quad 0 \leq \theta \leq \frac{\pi}{2}}$

4) $\iiint_E 3 \, dx\,dy\,dz$ ↙ $\frac{y}{3} = \frac{r\cos(\theta)}{3}$

$$= \int_0^{\pi/2} \int_0^3 \int_0^{\frac{r\cos(\theta)}{3}} 3 \quad r \, dx \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^3 \cancel{3} r \left(\frac{r\cos(\theta)}{\cancel{3}} \right) dr \, d\theta$$

$$= \left(\int_0^3 r^2 dr \right) \left(\int_0^{\pi/2} \cos(\theta) \, d\theta \right)$$

$$= \left[\frac{r^3}{3} \right]_0^3 \left[\sin(\theta) \right]_0^{\pi/2}$$

$$= \left(\frac{27}{3} \right)$$

$$= 9$$

5. *(10 points)* Find the (absolute value) of the Jacobian of the following change of variables

$$\begin{cases} x = \rho \sin(\phi)\cos(\theta) \\ y = \rho \sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases}$$

$$\frac{dx\,dy\,dz}{d\rho\,d\theta\,d\phi} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{vmatrix}$$

$$= \cos(\phi)\begin{vmatrix} -\rho\sin(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \rho\sin(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) \end{vmatrix}$$

$$-\rho\sin(\phi)\begin{vmatrix} \sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \end{vmatrix}$$

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$$= \cos(\phi)\left(\underbrace{-\rho^2 \sin(\phi)\cos(\phi)\sin^2(\theta) - \rho^2 \sin(\phi)\cos(\phi)\cos^2(\theta)}_{-\rho^2 \sin(\phi)\cos(\phi)}\right)$$

$$-\rho \sin(\phi)\left(\underbrace{\rho \sin^2(\phi)\cos^2(\theta) + \rho \sin^2(\phi)\sin^2(\theta)}_{\rho \sin^2(\phi)}\right)$$

$$= -\rho^2 \sin(\phi)\cos^2(\phi) - \rho^2 \sin^3(\phi)$$

$$= -\rho^2 \sin(\phi)\left(\cos^2(\phi) + \sin^2(\phi)\right)$$

$$= -\rho^2 \sin(\phi)$$

HENCE THE JACOBIAN IS $\left|-\rho^2 \underbrace{\sin(\phi)}_{\geq 0}\right| = \boxed{\rho^2 \sin(\phi)}$

(SINCE $0 \leq \phi \leq \pi$)