MOCK FINAL

Instructions: This is a mock final, designed to give you some practice for the actual final. Although it will be similar in spirit to the actual final, beware that the final might have different questions, and possibly in a different order. So please also look at the study guide and the suggested homework for a more complete study experience!

1		10
2		10
3		10
4		10
5		15
6		5
7		10
8		15
9		15
Total		100

Note: The equations for spherical coordinates are $x = \rho \sin(\phi) \cos(\theta), y = \rho \sin(\phi) \sin(\theta), z = \rho \cos(\phi)$ and the Jacobian is $\rho^2 \sin(\phi)$.

Date: Friday, December 14, 2018.

1. *(10 points)* Evaluate the following integral, where E is the region above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2+y^2+z^2 = 4$:

$$\iiint_E z \, dxdydz$$

2. *(10 points)* Use the following change of variables to evaluate

$$\iint_D \cos\left(2x^2 + 4y^2\right) dxdy$$

where D is the region in the first quadrant bounded by the ellipse $x^2 + 2y^2 \leq 1$:

$$\begin{aligned} x =& r\cos(\theta) \\ y =& \frac{r}{\sqrt{2}} \sin(\theta) \end{aligned}$$

3. *(10 points)* Find the area of the region D that is bounded by the curve C given by the following parametric equations, where $0 \leq t \leq 2\pi$

$$x(t) = 3\cos(t) - 2\sin(t)$$
$$y(t) = 3\cos(t) + 2\sin(t)$$

4. *(10 points)* Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle yze^{xz}, e^{xz}, xye^{xz} \rangle$ and $\mathbf{r}(t) = \langle t^2+1, t^2-1, t^2-2t \rangle$, and $0 \le t \le 2$. Justify all your steps.

5. *(15 points, 5 points each)* Let S be the helicoid given by the parametric equations $\mathbf{r}(u, v) = \langle u\cos(v), u\sin(v), v\rangle$, $0 \leq u \leq 1, 0 \leq v \leq \frac{\pi}{2}$.

   (a) Find the equation of the tangent plane to the surface at the point $(x, y, z) = \left(1, 1, \frac{\pi}{4}\right)$

(b) Find $\iint_S \sqrt{x^2 + y^2} dS$

(c) Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle zx, zy, xy \rangle$

6. *(5 points)* Is there a vector field $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl}\mathbf{G}$, where $\mathbf{F}(x, y, z) = \langle xz, xyz, -y^2\rangle$?

7. *(10 points)* Find $\iint_S curl(\mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle -y, x, -2 \rangle$, and S is the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 4$, oriented in such a way that the boundary curve is counterclockwise

8. *(15 points)* Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle z \tan^{-1}(y^2), z^3 \ln\left(x^2+1\right), z \rangle$, and S is the part of the paraboloid $z = 2 - x^2 - y^2$ that lies *strictly* above the plane $z = 1$, oriented upward.

   Warning: S is not closed!

9. *(15 points, 5 points each)* (Since it's awesome, and probably no one has done it) In this problem, we'll show that the surface area of a sphere is the derivative of the volume of the sphere!!! After this problem it shouldn't be surprising that $\left(\frac{4}{3}\pi r^3\right)' = 4\pi r^2$ or $\left(\pi r^2\right)' = 2\pi r$.

   (a) Let E be the ball centered at $(0, 0, 0)$ and radius R and let V be its volume. Using $V = \iiint_E 1\, dxdydz$ and the change of variables $u = \frac{x}{R}, v = \frac{y}{R}, z = \frac{z}{R}$, show that $V = CR^3$, where C is the volume of the ball centered at $(0, 0, 0)$ and radius 1.

(b) On the other hand, using:

(1) The divergence theorem with $\mathbf{F} = \frac{1}{3}\langle x, y, z\rangle$
(2) The 'adult' version of the surface integral (see Lecture 23)
(3) The formula for the unit normal vector $\mathbf{n}$ to a sphere of radius R

Show that $V = \left(\frac{R}{3}\right) S$, where S is the surface area of the sphere centered at $(0, 0, 0)$ and radius R.

(c) Use your answers from (a) and (b) to show that $S = 3CR^2$, and conclude that $V' = S$, so (in three dimensions) the derivative of the volume is the surface area.