MATH 409 – MOCK FINAL EXAM – SOLUTIONS

1. (a) Let $\epsilon > 0$ be given, let $\delta = \epsilon$, then if $|x| < \delta = \epsilon$, then

$$|f(x) - f(0)| = \left| x \sin\left(\frac{1}{x}\right) \right| = |x| \underbrace{\left| \sin\left(\frac{1}{x}\right) \right|}_{\leq 1} \leq |x| < \delta = \epsilon \checkmark$$

Therefore f is continuous at $x = 0$

(b)

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$

Since the limit on the right does not exist, it follows that f is not differentiable at 0

(c) Notice that, by part (a), $f(x)$ is continuous on $[0, 1]$, so f is a continuous extension of $\sin\left(\frac{1}{x}\right)$ to $[0, 1]$ and therefore from lecture, $\sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0, 1)$

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2. **Existence:** Consider $g(x) = f(x) - x$, which is continuous since f is continuous. Then $g(0) = f(0) - 0 = f(0) \geq 0$ (since $0 \leq f \leq 1$) and $g(1) = f(1) - 1 \leq 0$ (since $0 \leq f \leq 1$), therefore by the Intermediate Value Theorem, there is c in $(0, 1)$ such that $g(c) = 0$, that is $f(c) - c = 0$ so $f(c) = c$ ✓

Uniqueness: Suppose f has two fixed points a and b, that is $f(a) = a$ and $f(b) = b$, then by the Mean Value theorem, there is c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1$$

Which contradicts that $f'(x)$ is never 1 $\Rightarrow\Leftarrow$ ✓

3. (a) $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$

STEP 1: Scratchwork

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = |y - x|\, xy = |x - y|\, |x|\, |y|$$

However, $x \geq \frac{1}{2}$ so $|x| \geq \frac{1}{2}$ so $\frac{1}{|x|} \leq 2$ and similarly $\frac{1}{|y|} \leq 2$ and therefore

$$|f(x) - f(y)| = \frac{|x - y|}{|x|\, |y|} \leq 4\, |x - y| < \epsilon$$

Which gives $|x - y| < \frac{\epsilon}{4}$ so $\delta = \frac{\epsilon}{4}$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{4}$, then if $|x - y| < \delta$, then

$$|f(x) - f(y)| = |x - y|\, |x|\, |y| \leq 4\, |x - y| < 4\left(\frac{\epsilon}{4}\right) = \epsilon \checkmark$$

Hence f is uniformly continuous on $[\frac{1}{2}, \infty)$ □

(b) $f(x) = \sin(x)$ on $[0, 1]$

Since f is continuous on $[0, 1]$ and $[0, 1]$ is compact (closed and bounded), f is uniformly continuous on $[0, 1]$

(c) $f(x) = \frac{\sin(x)}{x}$ on $(0, 1]$

Since $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$,

$$\tilde{f}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is a continuous extension of f on $[0,1]$, and therefore f is uniformly continuous on $(0,1]$

(d) $f(x) = \frac{1}{x-3}$ on $(0,3)$

f is not uniformly continuous because it is not bounded on $(0,3)$: For every $M > 0$, notice $\frac{1}{x-3} > M$ implies $x - 3 < \frac{1}{M}$ which is true whenever $x < 3 + \frac{1}{M}$

(e) $f(x) = \sin(x)$ on $\mathbb{R}$

Notice $|f'(x)| = |\cos(x)| \leq 1$, therefore f is uniformly continuous on $\mathbb{R}$

4. (a) $\lim_{x\to 2} \frac{1}{(x-2)^2} = \infty$

We need to show that for all $M > 0$ there is δ such that if $0 < |x-2| < \delta$, then $\frac{1}{(x-2)^2} > M$

STEP 1: Scratch Work

$$\frac{1}{(x-2)^2} > M \Rightarrow (x-2)^2 < \frac{1}{M} \Rightarrow -\frac{1}{\sqrt{M}} < x-2 < \frac{1}{\sqrt{M}} \Rightarrow |x-2| < \frac{1}{\sqrt{M}}$$

STEP 2: Actual Proof:

Let $M > 0$ be given, let $\delta = \frac{1}{\sqrt{M}}$, then if $0 < |x-2| < \delta$, then

$$\frac{1}{(x-2)^2} > \frac{1}{\delta^2} = \frac{1}{\left(\frac{1}{\sqrt{M}}\right)^2} = M \checkmark$$

Therefore $\lim_{x\to 2} \frac{1}{(x-2)^2} = \infty$

(b) $\lim_{x\to 3^-} \sqrt{3-x} + 2 = 2$

We need to show that for all $\epsilon > 0$ there is δ such that if $0 < 3-x < \delta$ then $\left|\sqrt{3-x} + 2 - 2\right| < \epsilon$

STEP 1: Scratch Work

$$\left|\sqrt{3-x} + 2 - 2\right| = \sqrt{3-x} < \epsilon \Rightarrow 3-x < \epsilon^2$$

STEP 2: Actual Proof:

Let $\epsilon > 0$ be given, let $\delta = \epsilon^2$, then if $0 < 3 - x < \delta$, then

$$\left|\sqrt{3-x}+2-2\right| = \sqrt{3-x} < \sqrt{\delta} = \sqrt{\epsilon^2} = \epsilon \checkmark$$

Therefore $\lim_{x\to 3^-} \sqrt{3-x}+2 = 2$

(c) $\lim_{x\to -\infty} \frac{1}{x+2} = 0$

We need to show that for all $\epsilon > 0$ there is $N < 0$ such that if $x < N$ then $\left|\frac{1}{x+2} - 0\right| < \epsilon$

STEP 1: Scratch Work

$$\left|\frac{1}{x+2} - 0\right| = \frac{1}{|x+2|} < \epsilon \Rightarrow |x+2| > \frac{1}{\epsilon} \Rightarrow x+2 < -\frac{1}{\epsilon} \Rightarrow x < -2 - \frac{1}{\epsilon}$$

STEP 2: Actual Proof:

Let $\epsilon > 0$ be given, let $N = -2 - \frac{1}{\epsilon}$, then if $x < N$, then $x + 2 < -\frac{1}{\epsilon}$ so $|x+2| < \frac{1}{\epsilon}$ and so

$$\left|\frac{1}{x+2} - 0\right| = \frac{1}{|x+2|} < \frac{1}{\frac{1}{\epsilon}} = \epsilon \checkmark$$

Therefore

$$\lim_{x\to -\infty} \frac{1}{x+2} = 0$$

5. $U(f)$: Let P be the evenly spaced Calculus partition with $t_k = \frac{k}{n}$. Since x^2 is increasing, notice that:

$$M(f,[t_{k-1},t_k]) = f(t_k) = (t_k)^2 \quad \text{(Since } t_k \text{ is rational)}$$

$$\begin{aligned}
U(f,P) &= \sum_{k=1}^{n} M(f,[t_{k-1},t_k])\,(t_k - t_{k-1}) \\
&= \sum_{k=1}^{n} (t_k)^2\,(t_k - t_{k-1}) \\
&= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \left(\frac{1}{n}\right) \\
&= \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\
&= \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{(n+1)(2n+1)}{6n^2}
\end{aligned}$$

Since $U(f)$ is the inf over all partitions, we must have

$$U(f) \leq U(f,P) = \frac{(n+1)(2n+1)}{6n^2}$$

Therefore, taking the limit as $n \to \infty$ of the right hand side, we get $U(f) \leq \frac{2}{6} = \frac{1}{3}$.

$L(f)$**:** This is easier: For *any* partition P, we have $m(f,[t_{k-1},t_k]) = 0$, since any sub-piece $[t_{k-1},t_k]$ contains irrational numbers. Therefore

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} 0 = 0$$

And taking the sup over all partitions P we get

$$L(f) = \sup\{L(f,P) \mid P \text{ partition }\} = \sup\{0\} = 0$$

6. By the Mean Value Theorem for Integrals, there is c' in $(0,1)$ such that

$$\begin{aligned} f(c') &= \frac{\int_0^1 f(x)dx}{1-0} \\ &= \int_0^1 ax^3 + bx^2 + cx + ddx \\ &= \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_0^1 \\ &= \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d \\ &= 0 \text{ (By assumption)} \end{aligned}$$

Therefore there is c' such that $f(c') = 0$ $\square$

7. Since the statement is true for all g, we can let $g(x) = f(x)$, which is continuous, and so by assumption, we get

$$\int_a^b f(x)f(x)dx = 0 \Rightarrow \int_a^b \underbrace{(f(x))^2}_{\geq 0} dx = 0$$

Since $(f(x))^2$ is continuous (by assumption) and non-negative, from lecture we obtain that $(f(x))^2 = 0$ for all x and therefore $f(x) = 0$ for all x $\square$