

MATH 409 – MOCK FINAL EXAM

Note: This mock final is giving you more practice with the material after Midterm 2. Remember that the actual final is cumulative.

1. (10 points) Consider the following function:

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- (a) Use $\epsilon - \delta$ to show that f is continuous at 0
- (b) Show that f is not differentiable at 0
- (c) Use a theorem in lecture to show that $\sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0, 1)$

2. (*10 points*) Suppose $f : [0, 1] \rightarrow [0, 1]$ is differentiable and $f'(x) \neq 1$ for all x . Show that f has a unique fixed point.

3. (10 points) Show that the following functions are uniformly continuous. Unless stated otherwise, you may use any theorem covered in lecture

(a) $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$ (use $\epsilon - \delta$)

(b) $f(x) = \sin(x)$ on $[0, 1]$

(c) $f(x) = \frac{\sin(x)}{x}$ on $(0, 1]$

(d) $f(x) = \frac{1}{x-3}$ on $(0, 3)$

(e) $f(x) = \sin(x)$ on \mathbb{R}

4. (10 points) Use $\epsilon - \delta$ (or any variation thereof) to evaluate the following limits:

(a) $\lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = \infty$

(b) $\lim_{x \rightarrow 3^-} \sqrt{3-x} + 2 = 2$

(c) $\lim_{x \rightarrow -\infty} \frac{1}{x+2} = 0$

5. (10 points) For the following function f on $[0, 1]$, show that $U(f) \leq \frac{1}{3}$ and $L(f) = 0$

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Note: You may use $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

6. (10 points) Suppose $f(x) = ax^3 + bx^2 + cx + d$ is such that $\frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0$. Show that f has at least one root in $[0, 1]$

7. (10 points) Suppose f is continuous on $[a, b]$ and $\int_a^b f(x)g(x)dx = 0$ for every continuous g , show that $f(x) = 0$ for all x .