MATH 140A – MOCK FINAL EXAM – SOLUTIONS

1. Let $m = \inf(S)$, then

$$\inf(S) = -\sup(-S) \Leftrightarrow m = -\sup(-S) \Leftrightarrow -m = \sup(-S)$$

To show $-m = \sup(-S)$, we need to show that (1) $-m$ is an upper bound of $-S$ and (2) $-m$ is the least upper bound of $-S$

Upper Bound: Let $-s \in -S$, then, since $m = \inf(S)$, we have $m \leq s$, and so $-s \leq -m$. But since $-s$ was arbitrary in $-S$, $-m$ is an upper bound for $-S$ ✓

Least upper bound: Suppose $m_1 < -m$, we need to show that there is $-s \in -S$ such that $-s > m_1$. But since $m_1 < -m$, $-m_1 > m$, and so, since $m = \inf(S)$, there is $s \in S$ with $s < -m_1$, that is $-s > m_1$ ✓

Hence $-m = \sup(-S)$ and so

$$\inf(S) = m = -(-m) = -\sup(-S) \quad \square$$

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2. Let $\epsilon > 0$ be given.

Then, since $s_n \to s$, there is N_1 such that if $n > N_1$, then $|s_n - s| < \frac{\epsilon}{2}$.

And since $t_n \to t$, there is N_2 such that if $n > N_2$, then $|t_n - t| < \frac{\epsilon}{2}$

Let $N = \max\{N_1, N_2\}$, then if $n > N$ we have

$$\begin{aligned} |s_n - t_n - (s - t)| &= |s_n - s - t_n + t| \\ &= |s_n - s - (t_n - t)| \\ &\leq |s_n - s| + |t_n - t| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \checkmark \end{aligned}$$

Hence $s_n - t_n \to s - t$ $\square$

3. (a) Fix N

Claim:

$$\sup\{ks_n \mid n > N\} = k \sup\{s_n \mid n > N\}$$

Proof of Claim: Let $M = \sup\{s_n \mid n > N\}$, and let's show $\sup\{ks_n \mid n > N\} = kM$.

Upper Bound: If $n > N$, then, since M is an upper bound for $\{s_n \mid n > N\}$, $s_n \leq M$, and so, since $k \geq 0$, we have $ks_n \leq kM$. Therefore, kM is an upper bound for $\{ks_n \mid n > N\}$

Least Upper Bound: Suppose $M_1 < kM$, then $\frac{M_1}{k} < M$, and so by definition of M as a sup, there is $n > N$ such that $s_n > \frac{M_1}{k}$, that is $ks_n > M_1$ ✓(since ks_n is an element of $\{ks_n \mid n > N\}$)

Therefore we get

$$\sup\{ks_n \mid n > N\} = k \sup\{s_n \mid n > N\}$$

And taking limits, we have

$$\begin{aligned}
\limsup_{n\to\infty} ks_n &= \lim_{N\to\infty} \sup\{ks_n \mid n > N\} \\
&= \lim_{N\to\infty} k \sup\{s_n \mid n > N\} \\
&= k \lim_{N\to\infty} \sup\{s_n \mid n > N\} \\
&= k\left(\limsup_{n\to\infty} s_n\right) \checkmark
\end{aligned}$$

(b) **NO**: Let $s_n = (-1)^n$ and $k = -1$, then

$$\limsup_{n\to\infty} k s_n = \limsup_{n\to\infty} (-1)^{n+1} = 1, \text{ but}$$

$$k\left(\limsup_{n\to\infty} s_n\right) = (-1)\limsup_{n\to\infty}(-1)^n = (-1)(1) = -1$$

4. (a) Let $\{U_\alpha\}$ be any collection of open subsets of S, and let U be their union. If $x \in U$, then x is in some U_α for some α. But since U_α is open, there is $r > 0$ such that $B(x,r) \subseteq U_\alpha \subseteq U$ ✓

(b) Let $U_1, \ldots, U_N$ be finitely many open subsets of S and let U be their intersection. If $x \in U$, then for every $n = 1, \ldots, N$, $x \in U_n$, and therefore, since U_n is open, there is $r_n > 0$ such that $B(x, r_n) \subseteq U_n$. Let $r =: \min\{r_1, \ldots, r_N\} > 0$. Then, for every n, $B(x,r) \subseteq B(x,r_n) \subseteq U_n$ and therefore $B(x,r) \subseteq U$ (by definition of intersection) ✓✓

(c) **NO** Let $U_n = (-n, n)$ where $n \in \mathbb{N}$. Then each U_n is open in $\mathbb{R}$, but the intersection of the U_n is $\{0\}$, which is not open ✓

5. Let $f(x) = \frac{1}{\sqrt{x}}$ and consider the partial sums

$$\begin{aligned} s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} =& 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \\ =& \text{ Area Rectangle 1 } + \text{ Area Rectangle 2} \\ & + \cdots + \text{ Area Rectangle n} \\ =& \text{ Sum of Areas of Rectangles} \end{aligned}$$

Where Rectangle 1 is the rectangle with base $[1, 2]$ and height 1, Rectangle 2 is the rectangle with base $[2, 3]$ and height $\frac{1}{\sqrt{2}}, \ldots$ and Rectangle n is the rectangle with base $[n, n+1]$ and height $\frac{1}{\sqrt{n}}$.

Since f is decreasing on $[1, \infty)$, the sum of the areas of the rectangles is greater than or equal to the area under f from 1 to $n+1$, and hence

$$\begin{aligned} s_n \geq & \int_1^{n+1} \frac{1}{\sqrt{x}} dx \\ =& \left[2\sqrt{x}\right]_1^{n+1} \\ =& 2\sqrt{n+1} - 2 \end{aligned}$$

But since $\lim_{n\to\infty} 2\sqrt{n+1} - 2 = \infty$, by comparison, we get

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \lim_{n\to\infty} s_n = \infty \checkmark \quad \square$$

6. **STEP 1:** Notice that

$$f(x^4) = f\left(\left(x^2\right)^2\right) = f(x^2) = f(x)$$

And more generally, let's prove:

Claim: $f\left(x^{2^n}\right) = f(x)$ for all $n \geq 0$

Proof of Claim: Let P_n be the proposition $f\left(x^{2^n}\right) = f(x)$

Base Case: $(n = 0)$, then $f\left(x^{2^0}\right) = f(x^1) = f(x)$ ✓

Inductive Step: Suppose P_n is true, that is $f\left(x^{2^n}\right) = f(x)$, show P_{n+1} is true, that is $f\left(x^{2^{n+1}}\right) = f(x)$. But

$$f\left(x^{2^{n+1}}\right) = f\left(x^{2\times 2^n}\right) = f\left(\left(x^{2^n}\right)^2\right) = f\left(x^{2^n}\right) = f(x) \checkmark$$

(where, in the last step, we used the inductive hypothesis)

Hence P_{n+1} is true, and hence P_n is true for all n ✓

STEP 2: Fix $x \in (-1, 1)$ and let $s_n = x^{2^n}$. Notice that, if $n \to \infty$, $2^n \to \infty$, and hence, since $x \in (-1, 1)$, we have $|x| < 1$ and hence $s_n = x^{2^n} \to 0$. Therefore, since f is continuous, we have $\lim_{n\to\infty} f(s_n) = f(0)$.

However, taking $n \to \infty$ in the identity $f(x^{2^n}) = f(x)$, we get:

$$f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} f(x^{2^n}) = \lim_{n\to\infty} f(s_n) = f(0) \checkmark$$

And therefore $f(x) = f(0)$ for all x $\square$

7. Let $y \geq 0$ be given and let $f(x) = x^2$ on $[0, \infty)$

Case 1: $y \leq 1$

Then $f(0) = 0 \leq y$ and $f(1) = 1 \geq y$, so, since f is continuous on $[0, 1]$, by the Intermediate Value Theorem, there is $x \in [0, 1]$ such that $f(x) = y$, that is $x^2 = y$ ✓

Case 2: $y \geq 1$

Then $f(0) = 0 \leq y$ and $f(y) = y^2 \geq y$ (since $y \geq 1$) and therefore, since f is continuous on $[0, y]$, by the Intermediate Value Theorem, there is $x \in [0, y]$ such that $f(x) = y$, that is $x^2 = y$ ✓

Uniqueness: Suppose $a^2 = y$ and $b^2 = y$ for some $a, b \geq 0$ then

$$a^2 - b^2 = (a - b)(a + b) = y - y = 0$$

Hence, either $a - b = 0$, so $a = b$ ✓, or $a + b = 0$, so $a = -b$, in which case $a = 0$ and $b = 0$ (since a and b are non-negative), in which case $a = b$ as well ✓ □

8. STEP 1: Scratch Work

Let $\epsilon > 0$ be TBA and let $\delta > 0$ be given

$$\left|x^2 - y^2\right| = |x-y|\,|x+y| = |x-y|\,(x+y) \overset{?}{\geq} \epsilon$$

Let $a = |x-y|$, then since $|x-y| < \delta$, we get $\boxed{a < \delta}$

WLOG, assume $x < y$, then $|x-y| = y - x$

$$|x-y| = a \Rightarrow y - x = a \Rightarrow \boxed{\text{y = x + a}}$$

Finally, we get

$$|x-y|\,|x+y| = a\,(x + (x+a)) = a\,(2x+a) \geq \epsilon$$

Which gives

$$2x + a \geq \frac{\epsilon}{a} \Rightarrow 2x \geq \frac{\epsilon}{a} - a \Rightarrow 2x \geq \frac{\epsilon - a^2}{a} \Rightarrow x \geq \frac{\epsilon - a^2}{2a}$$

Now, in order to guarantee $x \geq 0$, we just need $\epsilon - a^2 \geq 0$, so $a^2 \leq \epsilon$ and so $\boxed{a \leq \sqrt{\epsilon}}$

Therefore let $x = \frac{\epsilon - a^2}{2a} \geq 0$ and

$$y = x + a = \frac{\epsilon - a^2}{2a} + a = \frac{\epsilon - a^2 + 2a^2}{2a} = \frac{\epsilon + a^2}{2a} \geq 0$$

STEP 2: Actual Proof:

Let $\epsilon > 0$ be anything you want

Let $\delta > 0$ be given and suppose $a < \min\{\delta, \sqrt{\epsilon}\}$.

Then let

$$x = \frac{\epsilon - a^2}{2a} \qquad y = x + a = \frac{\epsilon + a^2}{2a}$$

Then $x, y \in [0, \infty)$ and $|x - y| = |x - (x + a)| = |-a| = a < \delta$ but

$$|f(x) - f(y)| = |x - y|\,(x+y) = a\,(2x + a) = a\left(\frac{\epsilon}{a} - a + a\right) = a\left(\frac{\epsilon}{a}\right) = \epsilon \geq \epsilon\checkmark$$

Hence $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$ $\square$