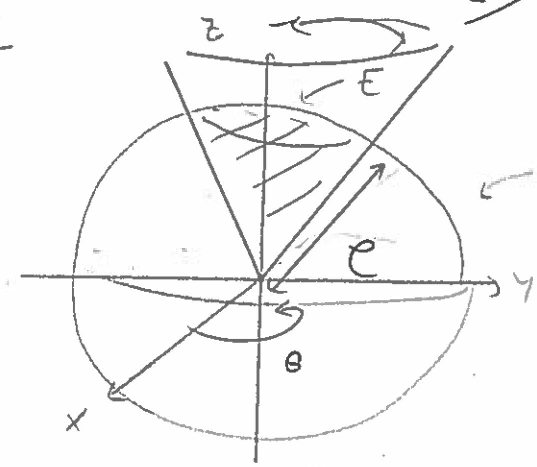
SOLUTIONS

1. *(10 points)* Evaluate the following integral, where E is the region above the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = 4$:

$$\iiint_E z \, dx dy dz$$

1) PICTURE

$\phi \leftarrow \cancel{z = \sqrt{x^2+y^2}}$ $\quad \phi = \pi/4$



$\cancel{x^2+y^2+z^2=4}$

$\rho = 2$

2)

$0 \leq \rho \leq 2$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi/4$

3) $\text{ANS} = \int_0^{\pi/4} \int_0^{2\pi} \int_0^2 \rho \cos(\phi) \, \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi$

$= \left(\int_0^2 \rho^3 \, d\rho\right) \left(\int_0^{2\pi} d\theta\right) \left(\int_0^{\pi/4} \underbrace{\cos(\phi)\sin(\phi)}_{\frac{1}{2}\sin(2\phi)} \, d\phi\right)$

$= \left[\frac{1}{4}\rho^4\right]_0^2 (2\pi) \left[\frac{-1}{4}\cos(2\phi)\right]_0^{\pi/4}$

$= \cancel{\left(\frac{1}{4}\right)}\cancel{(16)}(2\pi)\left(\frac{1}{4}\right)$

$= 2\pi$

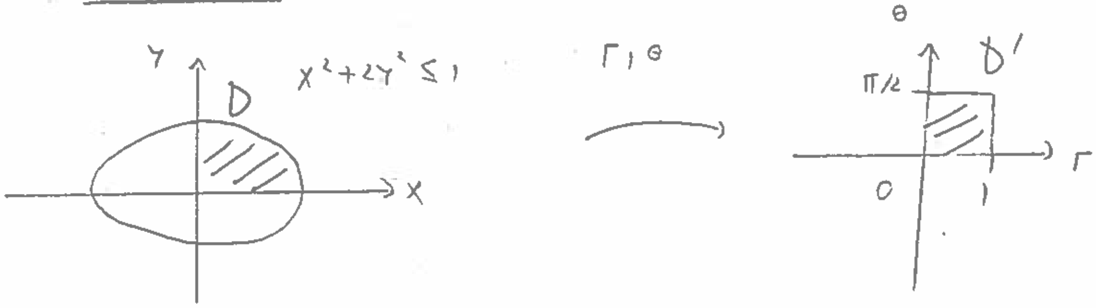
2. *(10 points)* Use the following change of variables to evaluate

$$\iint_D \cos\left(2x^2 + 4y^2\right) dx dy$$

where D is the region in the first quadrant bounded by the ellipse $x^2 + 2y^2 \leq 1$:

$$x = r\cos(\theta)$$
$$y = \frac{r}{\sqrt{2}}\sin(\theta)$$

1) FIND D'



NOTE $\quad X^2 + 2Y^2 \leq 1 \Rightarrow$

$$r^2\cos^2(\theta) + \cancel{2}\frac{r^2}{\cancel{2}}\sin^2(\theta) \leq 1$$

$$r^2\cos^2(\theta) + r^2\sin^2(\theta) \leq 1$$

$$r^2 \leq 1$$

$$r \leq 1, \text{ so } \quad 0 \leq r \leq 1$$

AND SINCE WE'RE IN THE FIRST QUADRANT, $\quad 0 \leq \theta \leq \frac{\pi}{2}$

2) JACOBIAN

$$dX\,dY = \left|\frac{dx\,dy}{dr\,d\theta}\right| dr\,d\theta = \left|\frac{r}{\sqrt{2}}\right| dr\,d\theta = \frac{r}{\sqrt{2}}\,dr\,d\theta$$

$$\frac{dx\,dy}{dr\,d\theta} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \frac{1}{\sqrt{2}}\sin(\theta) & \frac{r}{\sqrt{2}}\cos(\theta) \end{vmatrix} = \frac{r}{\sqrt{2}}\cos^2(\theta) + \frac{r}{\sqrt{2}}\sin^2(\theta)$$

$$= \frac{r}{\sqrt{2}}$$

FINALLY, $\cos(2x^2+4y^2)$

$$= \cos\left(2r^2\cos^2(\theta) + 4\frac{r^2}{2}\sin^2(\theta)\right)$$

$$= \cos(2r^2\cos^2(\theta) + 2r^2\sin^2(\theta))$$

$$= \cos(2r^2)$$

3)- SO $\iint_D \cos(2x^2+4y^2)\,dx\,dy$

$$= \iint_{D'} \cos(2r^2)\frac{r}{\sqrt{2}}\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 \cos(2r^2)\frac{r}{\sqrt{2}}\,dr\,d\theta$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}\right)\int_0^1 \cos(2r^2)\,r\,dr$$

$$= \frac{\pi}{2\sqrt{2}}\left[\frac{\sin(2r^2)}{4}\right]_0^1$$

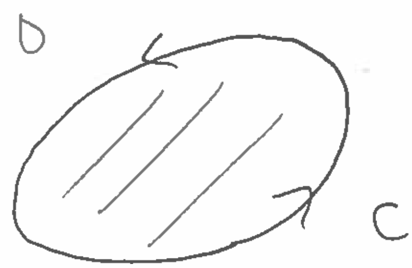
$$= \frac{\pi}{2\sqrt{2}}\,\frac{\sin(2)}{4}$$

$$= \frac{\pi}{8\sqrt{2}}\sin(2)$$

3. *(10 points)* Find the area of the region D that is bounded by the curve C given by the following parametric equations, where $0 \leq t \leq 2\pi$

$$x(t) = 3\cos(t) - 2\sin(t)$$
$$y(t) = 3\cos(t) + 2\sin(t)$$

1) PICTURE (NOT NECESSARY, JUST FOR VISUALIZATION)



(NOTE THE ORIENTATION IS OK SINCE D IS ON YOUR LEFT, BUT YOU DON'T HAVE TO WORRY ABOUT THAT)

2) $\text{AREA} = \frac{1}{2}\int_C x\,dy - y\,dx$

$$= \frac{1}{2}\int_0^{2\pi} x(t)\,y'(t) - y(t)\,x'(t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} (3\cos(t) - 2\sin(t))(-3\sin(t) + 2\cos(t)) - (3\cos(t) + 2\sin(t))(-3\sin(t) - 2\cos(t))\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} -\cancel{9\cos(t)\sin(t)} + 6\cos^2(t) + 6\sin^2(t) - \cancel{4\sin(t)\cos(t)} + \cancel{9\cos(t)\sin(t)} + 6\cos^2(t) + 6\sin^2(t) + \cancel{4\sin(t)\cos(t)}\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} 6 + 6\,dt = \left(\frac{1}{2}\right)(12)(2\pi) = 12\pi$$

4. *(10 points)* Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle yze^{xz}, e^{xz}, xye^{xz}\rangle$ and $\mathbf{r}(t) = \langle t^2+1, t^2-1, t^2-2t\rangle$, and $0 \le t \le 2$. Justify all your steps.

1) PICTURE

$r(2) = (5,3,0)$

$r(0) = (1,-1,0)$

SINCE C IS NOT CLOSED, IT MIGHT BE A GOOD IDEA TO USE THE FTC FOR LINE INTEGRALS

2) CHECK F IS CONSERVATIVE

$$\text{CURL}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yze^{xz} & e^{xz} & xye^{xz} \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(xye^{xz}) - \frac{\partial}{\partial z}(e^{xz}),\ -\frac{\partial}{\partial x}(xye^{xz}) + \frac{\partial}{\partial z}(yze^{xz}),\ \frac{\partial}{\partial x}(e^{xz}) - \frac{\partial}{\partial y}(yze^{xz}) \right\rangle$$

$$= \left\langle \cancel{xe^{xz}} - \cancel{xe^{xz}},\ -\cancel{ye^{xz}} - \cancel{xyze^{xz}} + \cancel{ye^{xz}} + \cancel{yzxe^{xz}},\ \cancel{ze^{xz}} - \cancel{ze^{xz}} \right\rangle$$

$$= \langle 0,0,0\rangle \ \checkmark$$

3) $F = \nabla f \Rightarrow \langle yze^{xz}, e^{xz}, xye^{xz}\rangle = \langle f_x, f_y, f_z\rangle$

$f_x = yze^{xz} \Rightarrow f = y\cancel{z}\,\frac{e^{xz}}{\cancel{z}} + \text{JUNK} = ye^{xz} + \text{JUNK}$

$f_y = e^{xz} \Rightarrow f = ye^{xz} + JUNK$

$f_z = xye^{xz} \Rightarrow f = \frac{\cancel{x}ye^{xz}}{\cancel{x}} + JUNK = ye^{xz} + JUNK$

$\Rightarrow \quad f(x,y,z) = ye^{xz}$

4) so $\int_C F \cdot dr = f(5,3,0) - f(1,-1,0)$

$= 3e^{5(0)} - (-1)e^{(1)(0)}$

$= 3+1$

$= 4$

5. *(15 points, 5 points each)* Let S be the helicoid given by the parametric equations $\mathbf{r}(u,v) = \langle u\cos(v), u\sin(v), v\rangle$, $0 \le u \le 1, 0 \le v \le \frac{\pi}{2}$.

(a) Find the equation of the tangent plane to the surface at the point $(x,y,z) = (1,1,\frac{\pi}{4})$

1) FIND U & V

$\langle U\cos(V), U\sin(V), V\rangle = \langle 1, 1, \pi/4\rangle$

$\Rightarrow \quad U\cos(V) = 1, \quad U\sin(V) = 1, \quad V = \frac{\pi}{4}$

$U\cos(\pi/4) = 1 \Rightarrow U\left(\frac{1}{\sqrt{2}}\right) = 1 \Rightarrow U = \sqrt{2}$

2) $r_U = \langle \cos(V), \sin(V), 0\rangle$

$r_V = \langle -U\sin(V), U\cos(V), 1\rangle$

3) $\hat{N} = r_U \times r_V = \begin{vmatrix} i & j & k \\ \cos(V) & \sin(V) & 0 \\ -U\sin(V) & U\cos(V) & 1 \end{vmatrix}$

$= \langle \sin(V), -\cos(V), U\cos^2(V) + U\sin^2(V)\rangle$

$= \langle \sin(V), -\cos(V), U\rangle$

4) AT $U = \sqrt{2}, V = \pi/4$, $\hat{N} = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \sqrt{2}\rangle$

5) POINT $(1, 1, \pi/4)$, $\hat{N} = \langle \frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, \sqrt{2}\rangle$

ANS $\boxed{\frac{\sqrt{2}}{2}(x-1) - \frac{\sqrt{2}}{2}(y-1) + \sqrt{2}(z - \pi/4) = 0}$

(b) Find $\iint_S \sqrt{x^2+y^2}\,dS$

FOUND $r_U \times r_V = \langle \sin(V), -\cos(V), U \rangle$

$$\|r_U \times r_V\| = \sqrt{\sin^2(V)+\cos^2(V)+U^2} = \sqrt{1+U^2}$$

so $$\iint_S \sqrt{x^2+y^2}\,dS = \iint_S \sqrt{(U\cos(V))^2+(U\sin(V))^2}\ \|r_U \times r_V\|\,dU\ldots$$

$$= \int_0^{\pi/2}\int_0^1 \sqrt{U^2}\ \sqrt{1+U^2}\ dU\,dV$$

$$= \int_0^{\pi/2}\int_0^1 U\sqrt{1+U^2}\ dU\,dV$$

$$= \frac{\pi}{2}\left[\frac{1}{\cancel{2}}\frac{\cancel{2}}{3}(1+U^2)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{\pi}{2}\left(\frac{1}{3}\right)\left(2^{\frac{3}{2}}-1^{\frac{3}{2}}\right)$$

$$= \boxed{\frac{\pi}{6}\left(2^{\frac{3}{2}}-1\right)} \qquad \left(=\frac{\pi}{6}\left(2\sqrt{2}-1\right)\right)$$

(c) Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle zx, zy, xy \rangle$

FOUND $\hat{N} = r_U \times r_V = \langle \sin(V), -\cos(V), \underbrace{U}_{\geq 0} \rangle$ ✓

$$\iint_S F \cdot d\vec{S} = \int_0^{\pi/2}\int_0^1 \langle \overset{z}{V}\, \overset{x}{U\cos(V)}, \overset{z}{V}\, \overset{y}{U\sin(V)}, \overset{x}{U\cos(V)}\overset{y}{U\sin(V)} \rangle \cdot \langle \sin(V), -\cos(V), U \rangle \, dU\, dV$$

$$= \int_0^{\pi/2}\int_0^1 \cancel{VU\cos(V)\sin(V)} - \cancel{VU\sin(V)\cos(V)} + U^3\cos(V)\sin(V)\, dU\, dV$$

$$= \left(\int_0^1 U^3\, dU\right)\left(\int_0^{\pi/2} \underbrace{\cos(V)\sin(V)}_{\frac{1}{2}\sin(2V)}\, dV\right)$$

$$= \left[\frac{1}{4}U^4\right]_0^1 \left[-\frac{1}{4}\cos(2V)\right]_0^{\pi/2}$$

$$= \left(\frac{1}{4}\right)\left(\frac{1}{4} + \frac{1}{4}\right)$$

$$= \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{8}$$

6. *(5 points)* Is there a vector field $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl} \mathbf{G}$, where $\mathbf{F}(x, y, z) = \langle xz, xyz, -y^2 \rangle$?

No

IF $F = \text{CURL}(G)$, THEN

$\text{DIV}(F) = \underbrace{\text{DIV}(\text{CURL}}(G)) = 0$

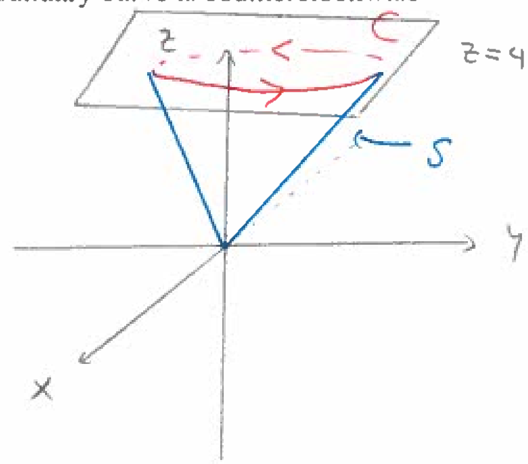
BUT $\text{DIV}(F) = (XZ)_X + (XYZ)_Y + (-Y^2)_Z$

$= Z + XZ \neq 0$

SO $0 \neq 0$ CONTRADICTION!

7. *(10 points)* Find $\iint_S curl(\mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle -y, x, -2\rangle$, and S is the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 4$, oriented in such a way that the boundary curve is counterclockwise

1) PICTURE



2) STOKES' THEOREM $\iint_S curl(F) \cdot d\vec{S} = \int_C F \cdot dr$

3) HERE C GOES COUNTERCLOCKWISE

NOTE $\underbrace{4}_{z} = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = 4^2 \Rightarrow$ C = CIRCLE OF RADIUS 4

$r(t) = \langle 4\cos(t), 4\sin(t), 4\rangle$

4) $\int_C F \cdot dr = \int_0^{2\pi} \langle \underbrace{-4\sin(t)}_{-y}, \underbrace{4\cos(t)}_{x}, \underbrace{-2}_{-2} \rangle \cdot \langle -4\sin(t), 4\cos(t), 0\rangle \, dt$

$= \int_0^{2\pi} \underbrace{16\sin^2(t) + 16\cos^2(t)}_{16} + 0 \, dt$

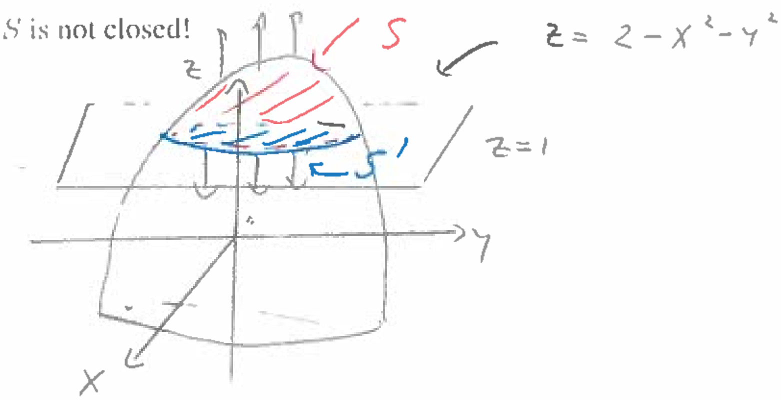
$= (16)(2\pi)$

$= 32\pi.$

8. *(15 points)* Calculate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle z\tan^{-1}(y^2), z^3\ln(x^2+1), z\rangle$, and S is the part of the paraboloid $z = 2 - x^2 - y^2$ that lies *strictly* above the plane $z = 1$, oriented upward.

Warning: S is not closed!

1) PICTURE



2) SINCE S IS NOT CLOSED, NEED TO CLOSE IT!

NOTE $\underbrace{1}_{z=1} = 2 - x^2 - y^2 \implies x^2 + y^2 = 1$

SO LET S' BE THE DISK OF RADIUS 1 IN THE PLANE $z=1$

(SEE PICTURE)

3) THEN $S + S'$ IS CLOSED, SO BY THE DIVERGENCE THEOREM,

$$\iint_{S+S'} F \cdot d\vec{S} = \iiint_E \operatorname{DIV}(F) = \iiint_E 1 \, dx\,dy\,dz$$

$$\operatorname{DIV}(F) = (z\tan^{-1}(y^2))_x + (z^3\ln(x^2+1))_y + (z)_z = 1$$

(TURN PAGE)

4) $\iiint_E 1 \, dx\,dy\,dz$

SMALL $\le z \le$ BIG

$1 \le z \le 2 - x^2 - y^2$

$1 \le z \le 2 - r^2$

D

D = DISK OF RADIUS 1

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$\iiint_E 1 \, dx\,dy\,dz = \int_0^{2\pi} \int_0^1 \int_1^{2-r^2} r \, dz\,dr\,d\theta$$

$$= 2\pi \int_0^1 (2 - r^2 - 1)\, r \, dr\,d\theta$$

$$= 2\pi \int_0^1 r - r^3 \, dr$$

$$= 2\pi \left[\frac{r^2}{2} - \frac{1}{4} r^4 \right]_0^1$$

$$= 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = 2\pi \left(\frac{1}{4} \right)$$

$$= \frac{\pi}{2}$$

5) Now $\iint_{S+S'} F \cdot d\vec{S} = \frac{\pi}{2}$

$$\underbrace{\iint_S F \cdot d\vec{S}}_{\text{WTF}} + \iint_{S'} F \cdot d\vec{S} = \frac{\pi}{2} \qquad (*)$$

6) $\iint_{S'} F \cdot d\vec{S}$ ⚠ S' NEEDS TO BE ORIENTED DOWNWARD

PARAMETRIZE S' : $r(x,y) = \langle x, y, 1 \rangle$

$r_x = \langle 1, 0, 0 \rangle$

$r_y = \langle 0, 1, 0 \rangle$

$$\hat{N} = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, \underbrace{1}_{>0} \rangle, \text{ USE } \langle 0, 0, -1 \rangle$$

$$\iint_{S'} F \cdot d\vec{S} = \iint_D \langle 1 \tan^{-1}(y^2), 1^3 \ln(x^2+1), 1 \rangle \cdot \langle 0, 0, -1 \rangle \, dx\,dy$$

$$= \int_0^{2\pi} \int_0^1 -1 \, r \, dr \, d\theta$$

$$= -2\pi \left[\frac{r^2}{2} \right]_0^1 = -2\pi \left(\frac{1}{2} \right) = -\pi$$

7) FROM (*): $\iint_S F \cdot d\vec{S} - \pi = \frac{\pi}{2} \Rightarrow \iint_S F \cdot d\vec{S} = \frac{3\pi}{2}$

9. *(15 points, 5 points each)* (Since it's awesome, and probably no one has done it) In this problem, we'll show that the surface area of a sphere is the derivative of the volume of the sphere!!! After this problem it shouldn't be surprising that $\left(\frac{4}{3}\pi r^3\right)' = 4\pi r^2$ or $(\pi r^2)' = 2\pi r$.

   (a) Let E be the ball centered at $(0,0,0)$ and radius R and let V be its volume. Using $V = \iiint_E 1\, dxdydz$ and the change of variables $u = \frac{x}{R}, v = \frac{y}{R}, W = \frac{z}{R}$, show that $V = CR^3$, where C is the volume of the ball centered at $(0,0,0)$ and radius 1.

1) $V = \iiint_E 1 \, dx\, dy\, dz$

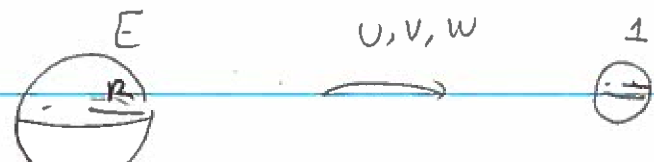
2) CHANGE OF VARS $\quad U = \frac{X}{R}, \; V = \frac{Y}{R}, \; W = \frac{Z}{R}$

THEN E BECOMES: $X^2 + Y^2 + Z^2 \leq R^2 \implies (RU)^2 + (RV)^2 + (RW)^2 \leq R^2$

$\implies \cancel{R^2} U^2 + \cancel{R^2} V^2 + \cancel{R^2} W^2 \leq \cancel{R^2}$

$\implies U^2 + V^2 + W^2 \leq 1$

SO E BECOMES $E' =$ BALL CENTERED AT $(0,0,0)$ AND RADIUS 1.



3) $dU\,dV\,dW = \left| \frac{dU\,dV\,dW}{dx\,dy\,dz} \right| dx\,dy\,dz = \left| \frac{1}{R^3} \right| dx\,dy\,dz = \frac{1}{R^3} dx\,dy\,dz$

$$\frac{dU\,dV\,dW}{dx\,dy\,dz} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{1}{R} & 0 & 0 \\ 0 & \frac{1}{R} & 0 \\ 0 & 0 & \frac{1}{R} \end{vmatrix} = \frac{1}{R}\frac{1}{R}\frac{1}{R} = \frac{1}{R^3}$$

SO $dx\,dy\,dz = R^3\, dU\,dV\,dW$

4) $V = \iiint_E 1 \, dx\,dy\,dz = \iiint_{E'} R^3\, dU\,dV\,dW = R^3 \iiint_{E'} 1\, dU\,dV\,dW = R^3 (C)$

$= CR^3$

(b) On the other hand, using:

(1) The divergence theorem with $\mathbf{F} = \frac{1}{3}\langle x, y, z\rangle$
(2) The 'adult' version of the surface integral (see Lecture 23)
(3) The formula for the unit normal vector $\mathbf{n}$ to a sphere of radius R

Show that $V = \left(\frac{R}{3}\right) S$, where S is the surface area of the sphere centered at $(0, 0, 0)$ and radius R.

$$V = \iiint_E 1 \, dx\,dy\,dz$$

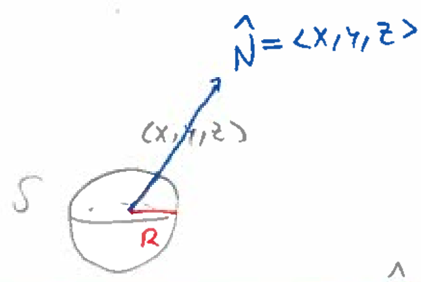
$$= \iiint_E \mathrm{DIV}(F)\, dx\,dy\,dz \quad \text{WHERE } F = \left\langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3} \right\rangle$$

(1)

$$= \iint_S F \cdot d\vec{S}$$

(2)

$$= \iint_S F \cdot N \, dS$$



$\hat{N} = \langle x, y, z\rangle, \quad N = \frac{\hat{N}}{\|\hat{N}\|} = \frac{1}{R}\langle x, y, z\rangle$

(3)

$$= \iint_S \frac{1}{3}\langle x, y, z\rangle \cdot \frac{1}{R}\langle x, y, z\rangle \, dS$$

$$= \iint_S \frac{1}{3R} \underbrace{(x^2 + y^2 + z^2)}_{R^2} \, dS$$

$$= \iint_S \frac{R^2}{3R}\, dS = \iint_S \frac{R}{3}\, dS = \frac{R}{3} \iint_S 1\, dS$$

$$= \frac{R}{3}\, \mathrm{AREA}(S) = \left(\frac{R}{3}\right) S$$

(c) Use your answers from (a) and (b) to show that $S = 3CR^2$, and conclude that $V' = S$, so (in three dimensions) the derivative of the volume is the surface area.

ON THE ONE HAND, BY (a),

$$V = CR^3$$

BUT ALSO $V = \frac{R}{3} S$ BY (b)

SO $CR^3 = \frac{R}{3} S$

$$\Rightarrow CR^2 = \frac{S}{3}$$

$$\Rightarrow \underline{S = 3CR^2}$$

AND SINCE $\underline{V = CR^3}$,

$$V' = (CR^3)' = 3CR^2 = S \quad ✓$$