SOLUTIONS

MOCK FINAL

PEYAM RYAN TABRIZIAN

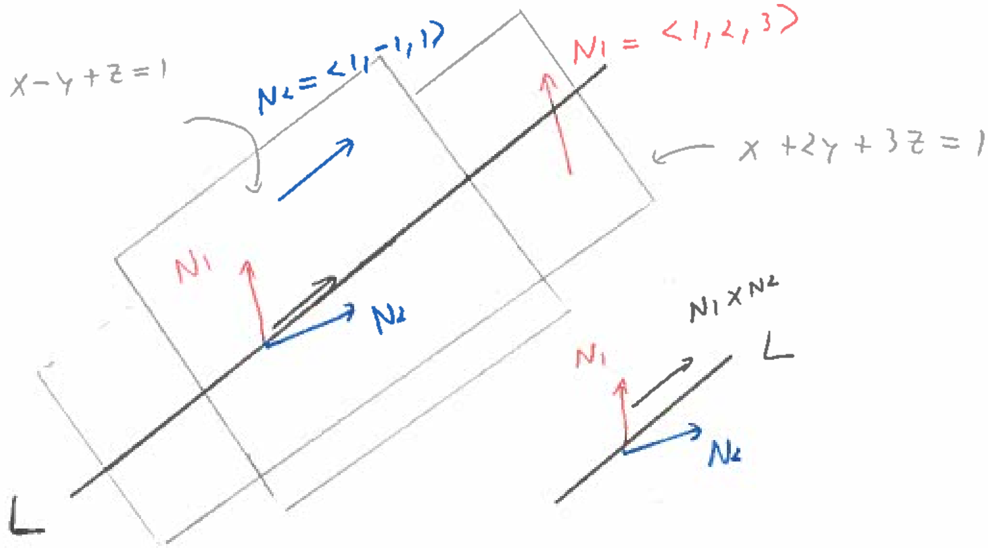
Instructions: This is a mock final, designed to give you some practice for the actual final. Do **NOT** expect the questions on the final to be the same; some will be easier, but most will be harder. Please also look at the study guide and the suggested homework for a more complete study experience!

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Date: Wednesday, June 13, 2018.

1. *(10 points)* Use normal vectors to find the line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$.

PICTURE



1) DIRECTION VECTOR

NOTICE THAT THE DIRECTION VECTOR OF L IS SIMPLY GIVEN BY A VECTOR PERPENDICULAR TO N_1 & N_2, BUT:

$$N_1 \times N_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} k$$

$$= 5i + 2j - 3k$$

$$= \langle 5, 2, -3 \rangle \leftarrow \text{DIRECTION VECTOR}$$

2) POINT FIND $\underline{\underline{A}}$ SOLUTION OF $\begin{cases} x + 2y + 3z = 1 \\ x - y + z = 1 \end{cases}$

FOR EXAMPLE, SET $\underline{z = 0}$, GET $\begin{cases} x + 2y = 1 \\ x - y = 1 \end{cases}$ $\Rightarrow 3y = 0$ (SUBTRACT)

$\Rightarrow \underline{y = 0}$

$\Rightarrow x + 2(0) = 1$ (FROM FIRST EQUATION)

$\Rightarrow \underline{x = 1}$

POINT $(1, 0, 0)$

$(1,0,0)$ $\langle 5, 2, -3 \rangle$

3) EQUATIONS

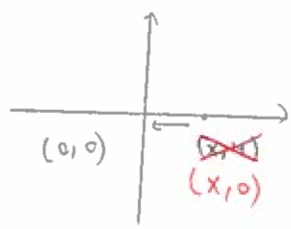
$$\boxed{\begin{cases} x(t) = 1 + 5t \\ y(t) = 2t \\ z(t) = -3t \end{cases}}$$

2. *(10 points)* Is the following function continuous at $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

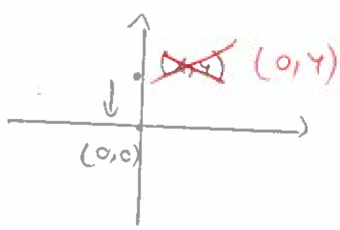
Is $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$?

ALONG X-AXIS



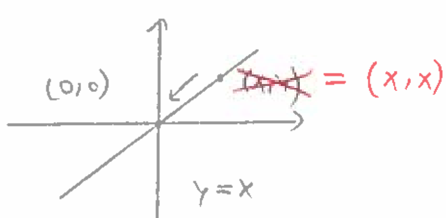
$y = 0$, so $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) \overset{y=0}{=} \lim_{x\to 0} \frac{x(0)}{x^2+x(0)+(0)^2} = \lim_{x\to 0}\frac{0}{x^2} = \boxed{0}$

ALONG Y-AXIS



$x = 0$, so $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) \overset{x=0}{=} \lim_{y\to 0} \frac{0(y)}{0^2+0(y)+y^2} = \lim_{y\to 0}\frac{0}{y^2} = \boxed{0}$

ALONG $y=x$



$y = x$, so $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) \overset{y=x}{=} \lim_{x\to 0} \frac{xx}{x^2+xx+x^2} = \lim_{x\to 0}\frac{\cancel{x^2}}{3\cancel{x^2}} = \boxed{\frac{1}{3}}$

SINCE WE GET TWO DIFFERENT LIMITS, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ DOES NOT EXIST,

SO f IS NOT CONTINUOUS AT $(0,0)$.

3. *(10 points)* Find an approximate value of

$$\sqrt{(4.2)^2 + (0.1)^2 + (2.9)^2}$$

LET $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

THE LINEARIZATION OF f ABOUT THE POINT $(4,0,3)$ IS:

$L(x,y,z) = f(4,0,3) + f_x(4,0,3)(x-4) + f_y(4,0,3)(y-0) + f_z(4,0,3)(z-3)$

$f(4,0,3) = \sqrt{4^2+0^2+3^2} = \sqrt{25} = 5$

$f_x(x,y,z) = \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}(\cancel{2}x) = \frac{x}{\sqrt{x^2+y^2+z^2}}$

$f_x(4,0,3) = \frac{4}{\sqrt{4^2+0^2+3^2}} = \frac{4}{5}$

$f_y(x,y,z) = \frac{1}{2\sqrt{x^2+y^2+z^2}} 2y = \frac{y}{\sqrt{x^2+y^2+z^2}}$

$f_y(4,0,3) = \frac{0}{\sqrt{4^2+0^2+3^2}} = 0$

$f_z(x,y,z) = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$f_z(4,0,3) = \frac{3}{\sqrt{4^2+0^2+3^2}} = \frac{3}{5}$

SO $L(x,y,z) = 5 + \frac{4}{5}(x-4) + 0(y) + \frac{3}{5}(z-3) = 5 + \frac{4}{5}(x-4) + \frac{3}{5}(z-3)$

SO $\sqrt{(4.2)^2+(0.1)^2+(2.9)^2} = f(4.2, 0.1, 2.9) \approx L(4.2, 0.1, 2.9)$

$= 5 + \frac{4}{5}(4.2-4) + \frac{3}{5}(2.9-3) = 5 + \frac{4}{5}(0.2) + \frac{3}{5}(-0.1) = 5 + (0.8)(0.2) + (0.6)(-0.1)$

$= 5 + 0.16 - 0.06$

$= 5.1$

4. *(10 points)* Find $\frac{\partial z}{\partial x}$ at $(0, 1)$ where $\ln(z) = xyz$

METHOD 1 DO IT DIRECTLY:

$$(\ln(z))_x = (xyz)_x$$

$$\frac{\frac{\partial z}{\partial x}}{z} = yz + xy\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x}\left(\frac{1}{z} - xy\right) = yz$$

$$\frac{\partial z}{\partial x} = \frac{yz}{\frac{1}{z} - xy}$$

NOW $x = 0$, $y = 1$ AND TO FIND z, USE $\ln(z) = xyz$

$$\ln(z) = (0)(1)(z) = 0$$

$$\Rightarrow \ln(z) = 0 \Rightarrow z = 1$$

SO FOR $x = 0$, $y = 1$, $z = 1$ THIS BECOMES:

$$\frac{\partial z}{\partial x} = \frac{(1)(1)}{\frac{1}{1} - (0)(1)} = 1$$

METHOD 2 USE THE FORMULA: $\ln(z) = xyz \Rightarrow \underbrace{\ln(z) - xyz}_{F(x,y,z)} = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(\ln(z) - xyz)_x}{(\ln(z) - xyz)_z}$$

$$= \frac{-(-yz)}{\frac{1}{z} - xy} = \frac{yz}{\frac{1}{z} - xy}$$

AND THE REST IS LIKE METHOD 1.

5. *(10 points)* Show that the equation of the tangent plane to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at the point (x_0, y_0, z_0) can be written as:

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1$$

$$F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$$

THE TANGENT PLANE TO THE ELLIPSOID $F=0$ AT (x_0,y_0,z_0) HAS EQUATION:

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

CALCULATE F_x, F_y, F_z ↓

$$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0)=0$$

EXPAND OUT ↓

$$\frac{2x_0x}{a^2}-\frac{2x_0^2}{a^2}+\frac{2y_0y}{b^2}-\frac{2y_0^2}{b^2}+\frac{2z_0z}{c^2}-\frac{2z_0^2}{c^2}=0$$

$$\Longrightarrow \cancel{2}\frac{xx_0}{a^2}+\cancel{2}\frac{yy_0}{b^2}+\cancel{2}\frac{zz_0}{c^2}=\cancel{2}\frac{x_0^2}{a^2}+\cancel{2}\frac{y_0^2}{b^2}+\cancel{2}\frac{z_0^2}{c^2}$$

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=\underbrace{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}}$$

$= 1$ SINCE (x_0,y_0,z_0) IS ON THE ELLIPSOID AND HENCE SATISFIES THE EQUATION OF THE ELLIPSOID

$$\Longrightarrow \boxed{\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1}$$

6. *(10 points)* Find the local maximum and minimum values and saddle points of the function $f(x, y) = x^4 - 2x^2 + y^3 - 3y$.

1) CRITICAL POINTS

$f_x(x,y) = 4x^3 - 4x = 0 \Rightarrow 4x(x^2-1) = 0$

$\Rightarrow x=0$ OR $x=-1$ OR $x=1$

$f_y(x,y) = 3y^2 - 3 = 0 \Rightarrow \cancel{3}y^2 = \cancel{3} \Rightarrow y^2 = 1$

$\Rightarrow y = -1$ OR $y=1$

SO ($x=0$ OR $x=-1$ OR $x=1$) AND ($y=-1$ OR $y=1$)

THIS GIVES US 6 CRITICAL POINTS : $(0,-1), (0,1), (-1,-1), (-1,1), (1,-1), (1,1)$

2) $D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x^2-4 & 0 \\ 0 & 6y \end{vmatrix}$

(i) $D(0,-1) = \begin{vmatrix} -4 & 0 \\ 0 & -6 \end{vmatrix} = 24 > 0$ AND $f_{xx}(0,-1) = -4 < 0$

SO $f(0,-1) = -1+3 = 2$ IS A LOCAL MAX

(ii) $D(0,1) = \begin{vmatrix} -4 & 0 \\ 0 & 6 \end{vmatrix} = -24 < 0$, SO $(0,1)$ IS A SADDLE POINT

(iii) $D(-1,-1) = \begin{vmatrix} 8 & 0 \\ 0 & -6 \end{vmatrix} = -48 < 0$, SO $(-1,-1)$ IS A SADDLE POINT

(IV) $D(-1,1) = \begin{vmatrix} 8 & 0 \\ 0 & 6 \end{vmatrix} = 48 > 0$ AND $f_{xx}(-1,1) = 8 > 0$, SO

$f(-1,1) = 1-2+1-3 = -3$ IS A LOCAL MIN

(V) $D(1,-1) = \begin{vmatrix} 8 & 0 \\ 0 & -6 \end{vmatrix} = -48 < 0$, SO $(1,-1)$ IS A SADDLE POINT

(vi) $D(1,1) = \begin{vmatrix} 8 & 0 \\ 0 & 6 \end{vmatrix}$

$= 48 > 0$, $f_{xx}(1,1) = 8$

SO $f(1,1) = 1-2+1-3$

$= -3$ IS A LOCAL MIN

7. *(10 points)* **Note:** This question has two parts to give you extra practice, but a more reasonable question about this on the final would only have one part.

(a) Use Lagrange multipliers to show that among all boxes with fixed volume V, the one with the smallest surface area must be a cube.

$$f(x,y,z) = 2xy + 2yz + 2xz$$

$$g(x,y,z) = xyz - V \qquad (\text{BECAUSE } xyz = \underbrace{V}_{\text{FIXED}})$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ \text{CONSTRAINT} \end{cases} \Rightarrow \begin{cases} 2y + 2z = \lambda yz \\ 2x + 2z = \lambda xz \\ 2x + 2y = \lambda xy \\ xyz = V \end{cases} \Rightarrow \begin{cases} \lambda = \dfrac{2(y+z)}{yz} \\ \quad\; \| \\ \lambda = \dfrac{2(x+z)}{xz} \\ \quad\; \| \\ \lambda = \dfrac{2(x+y)}{xy} \\ xyz = V \end{cases}$$

$$\frac{\cancel{2}(y+z)}{y\cancel{z}} \overset{\cancel{\lambda}}{=} \frac{\cancel{2}(x+z)}{x\cancel{z}} \Rightarrow x(y+z) = y(x+z)$$

$$\Rightarrow \cancel{xy} + xz = \cancel{xy} + yz$$

$$\Rightarrow x\cancel{z} = y\cancel{z}$$

$$\Rightarrow x = y$$

$$\frac{\cancel{2}(x+z)}{\cancel{x}z} = \frac{\cancel{2}(x+y)}{\cancel{x}y} \Rightarrow y(x+z) = z(x+y)$$

$$\Rightarrow xy + \cancel{yz} = xz + \cancel{yz}$$

$$\Rightarrow \cancel{x}y = \cancel{x}z$$

$$\Rightarrow y = z$$

HENCE $x = y$ & $y = z$, SO $x = y = z$ AND THE IDEAL BOX IS A CUBE.

(b) Use Lagrange multipliers to show that amoung all boxes with fixed surface area S, the one with the largest volume must be a cube.

$$f(x,y,z) = xyz$$

$$g(x,y,z) = 2xy + 2yz + 2xz - S$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ \text{CONSTRAINT} \end{cases} \Longrightarrow \begin{cases} yz = \lambda(2y+2z) \\ xz = \lambda(2x+2z) \\ xy = \lambda(2x+2y) \\ 2xy+2yz+2xz = S \end{cases}$$

$$\Longrightarrow \begin{cases} \lambda = \dfrac{yz}{2(y+z)} \\ \quad \parallel \\ \lambda = \dfrac{xz}{2(x+z)} \\ \quad \parallel \\ \lambda = \dfrac{xy}{2(x+y)} \\ 2xy+2yz+2xz = S \end{cases}$$

$$\frac{y\cancel{z}}{\cancel{2}(y+z)} = \frac{x\cancel{z}}{\cancel{2}(x+z)} \Longrightarrow y(x+z) = x(y+z)$$
$$\Longrightarrow \cancel{xy} + yz = \cancel{xy} + xz$$
$$\Longrightarrow y\cancel{z} = x\cancel{z} \Longrightarrow y = x$$

$$\frac{\cancel{x}z}{\cancel{2}(x+z)} = \frac{\cancel{x}y}{\cancel{2}(x+y)} \Longrightarrow z(x+y) = y(x+z)$$
$$\Longrightarrow xz + \cancel{yz} = xy + \cancel{yz}$$
$$\Longrightarrow \cancel{x}z = \cancel{x}y \Longrightarrow z = y$$

SO $y = x$ AND $z = y$, HENCE $x = y = z$ AND THE IDEAL BOX IS AGAIN A CUBE

8. *(10 points)* Calculate

$$\int_0^1 \int_{x^2}^1 \sqrt{y}\sin(y)\,dy\,dx$$

IMPOSSIBLE !

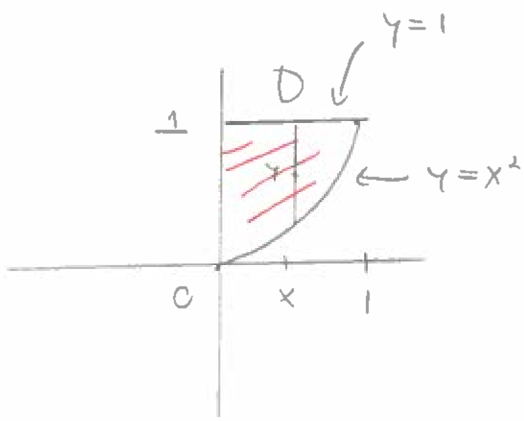
10

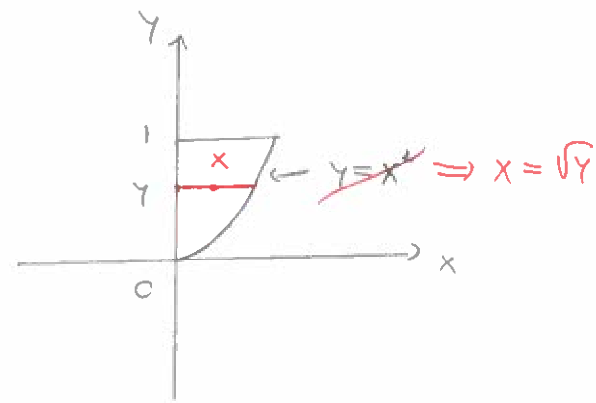
LET'S SWITCH THE ORDER OF dx & dy

1) DRAW D

KNOW $x^2 \leq y \leq 1$

$0 \leq x \leq 1$





2) WRITE D AS A HORIZONTAL REGION

LEFT $\leq x \leq$ RIGHT

$0 \leq x \leq \sqrt{y}$

$0 \leq y \leq 1$

3) SO OUR INTEGRAL BECOMES:

$$\int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\,\sin(y)\,dx\,dy = \int_0^1 \left[\sqrt{y}\,\sin(y)\,x\right]_{x=0}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \sqrt{y}\,\sin(y)\,\sqrt{y}\,dy = \int_0^1 y\,\sin(y)\,dy \longrightarrow \text{INTEGRATE BY PARTS (SORRY !)}$$

$$= \left[y(-\cos(y))\right]_0^1 - \int_0^1 (-\cos(y))\,dy = 1(-\cos(1)) - 0(-\cos(0)) + \left[\sin(y)\right]_0^1$$

$$= -\cos(1) + \sin(1) - 0$$

$$= \sin(1) - \cos(1)$$

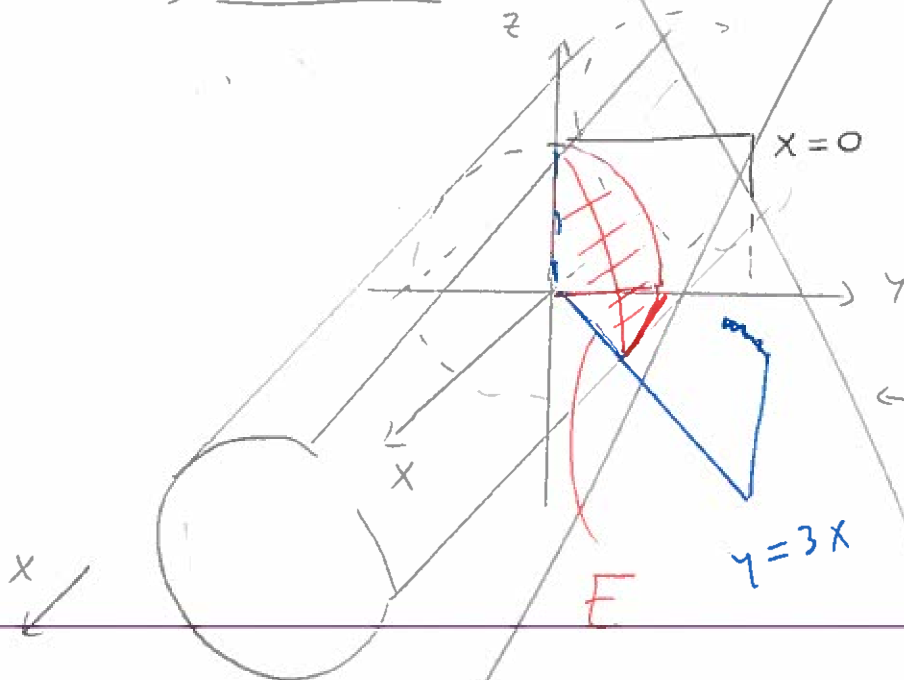
10. *(10 points)* Calculate

$$\iiint_E z\, dxdydz$$

where E is the solid in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$.

METHOD 1 $\longrightarrow$ IF YOU FIND METHOD 1 TOO DIFFICULT, SKIP TO METHOD 2.

1) DRAW E

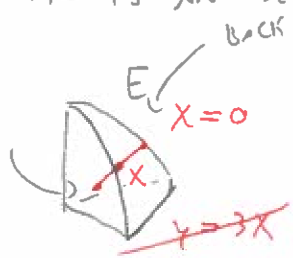


FIRST OCTANT

$y^2 + z^2 = 9$

2) NOTICE THAT THIS REGION IS FACING THE X-DIRECTION, SO IT IS AN X-REGION (THE BOOK CALLS THIS TYPE 2)

FRONT

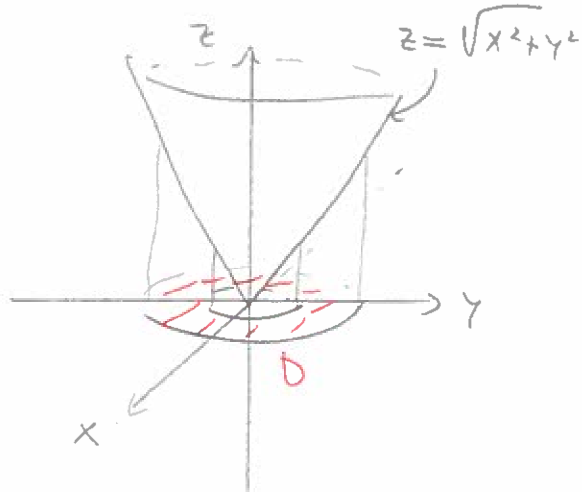


$\cancel{y = 3x} \Rightarrow x = \frac{y}{3}$

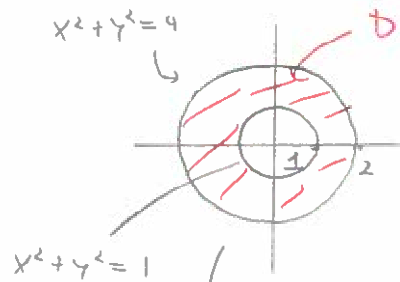
FRONT $\leq x \leq$ BACK $\Rightarrow$ $0 \leq x \leq \frac{y}{3}$

9. *(10 points)* Find the volume of the solid below the function $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leq x^2 + y^2 \leq 4$

1) PICTURE $\qquad z = \underbrace{\sqrt{x^2+y^2}}_{\geq 0} \Rightarrow z^2 = x^2 + y^2 \qquad$ CONE!



D = RING $1 \leq x^2+y^2 \leq 4$



NOTICE $\quad 1 \leq r \leq 2$, $\quad 0 \leq \theta \leq 2\pi$

2) $$V = \iint_D \sqrt{x^2+y^2}\, dx\, dy$$

$$= \int_0^{2\pi} \int_1^2 \underbrace{\sqrt{r^2}}_{r}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^2 r^2\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{3} r^3\right]_1^2 d\theta$$

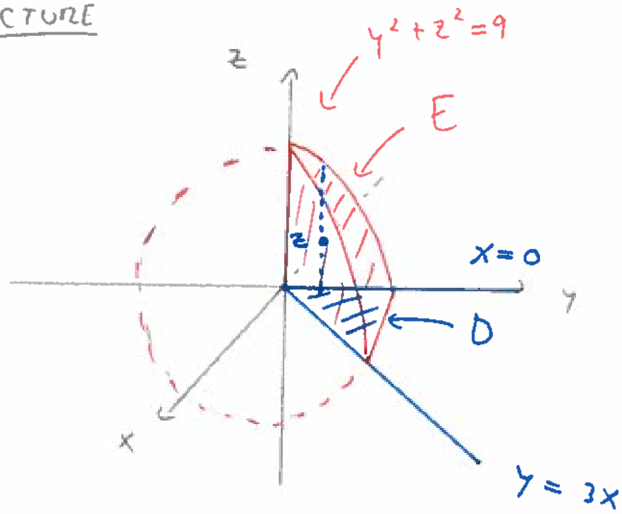
$$= \int_0^{2\pi} \frac{8}{3} - \frac{1}{3}\, d\theta = \int_0^{2\pi} \frac{7}{3}\, d\theta = \frac{7}{3}(2\pi) = \frac{14\pi}{3}$$

10. *(10 points)* Calculate

$$\iiint_E z\,dxdydz$$

where E is the solid in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$.

1) PICTURE



(NOTE TO HELP YOU DRAW THIS, FIRST PLOT THE LINES $X=0$ AND $Y=3X$ IN THE XY PLANE, THEN PLOT $y^2+z^2=9$ IN THE YZ-PLANE, AND NOTICE THAT ON THE PLANE $y=3X$, $y^2+z^2=9 \Rightarrow 9x^2+z^2=9$, WHICH IS AN ELLIPSE. FINALLY, CONNECT THE TWO SLICES YOU FOUND)

2) SMALLER $\leq z \leq$ BIGGER

BUT $y^2+z^2=9 \Rightarrow z=\sqrt{9-y^2}$ (SINCE $z \geq 0$),

SO BIGGER $= \sqrt{9-y^2}$

SMALLER $= 0$

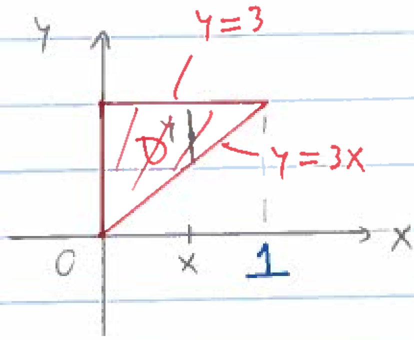
$$\boxed{0 \leq z \leq \sqrt{9-y^2}}$$

3) FIND D

NOTICE THAT $z=0$ IN THE XY-PLANE, SO $y^2+z^2=9 \Rightarrow y^2=9 \Rightarrow y=3$ (SINCE $y \geq 0$)

WHICH IS A LINE.

9. *(10 points)* Find the volume of the solid below the function $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leq x^2 + y^2 \leq 4$



SMALLER $\leq y \leq$ BIGGER

$$\boxed{3x \leq y \leq 3}$$

FINALLY, IF $3x = 3$, THEN $\underline{x = 1}$, SO $\boxed{0 \leq x \leq 1}$

4) HENCE $\displaystyle\iiint_E z \, dx\, dy\, dz$

$$= \int_0^1 \int_{3x}^{3} \int_0^{\sqrt{9-y^2}} z \, dz\, dy\, dx$$

$$= \int_0^1 \int_{3x}^{3} \left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{9-y^2}} dy\, dx$$

$$= \int_0^1 \int_{3x}^{3} \frac{(9-y^2)}{2} \, dy\, dx$$

$$= \int_0^1 \left[\frac{9}{2}y - \frac{y^3}{6}\right]_{y=3x}^{y=3} dx$$

$$= \int_0^1 \frac{27}{2} - \frac{27}{6} - \frac{27x}{2} + \frac{27x^3}{6} \, dx$$

$$= \int_0^1 9 - \frac{27}{2}x + \frac{9}{2}x^3 \, dx$$

$$= \left[9x - \frac{27}{4}x^2 + \frac{9}{8}x^4 \right]_0^1$$

$$= 9 - \frac{27}{4} + \frac{9}{8}$$

$$= \frac{72 - 54 + 9}{8}$$

$$= \frac{27}{8}$$