MATH 140A – MOCK MIDTERM 1 – SOLUTIONS

1.

STEP 1: First of all, if $s \in A + B$, then $s = a + b$ where $a \in A$ and $b \in B$, but by definition of $\sup(A)$ we get $a \leq \sup(A)$ and similarly $b \leq \sup(B)$, hence

$$s = a + b \leq \sup(A) + \sup(B)$$

Since s was arbitrary, $\sup(A) + \sup(B)$ is an upper bound for $A+B$, so because $\sup(A+B)$ is the *least* upper bound for $A+B$, we get

$$\sup(A + B) \leq \sup(A) + \sup(B) \checkmark$$

STEP 2: Fix $a \in A$, then for every $b \in B$, since $a + b \in A + B$ and by definition of $\sup(A + B)$, we get:

$$\begin{aligned} a + b &\leq \sup(A + B) \\ a &\leq \sup(A + B) - b \end{aligned}$$

But since $a \in A$ is arbitrary, $\sup(A + B) - b$ is an upper bound for A, and hence since $\sup(A)$ is the *least* upper bound:

$$\begin{aligned} \sup(A) &\leq \sup(A + B) - b \\ b &\leq \sup(A + B) - \sup(A) \end{aligned}$$

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But since $b \in B$ is arbitrary, $\sup(A+B) - \sup(A)$ is an upper bound for B, so since $\sup(B)$ is the *least* upper bound:

$$\begin{aligned} \sup(B) &\leq \sup(A+B) - \sup(A) \\ \sup(A) + \sup(B) &\leq \sup(A+B) \checkmark \end{aligned}$$

Therefore $\sup(A+B) = \sup(A) + \sup(B)$. $\square$

2.

STEP 1: Scratchwork

Since (s_n) converges, (s_n) is bounded above, so there is $M > 0$ such that $|s_n| \leq M$ for all n.,

$$\begin{aligned}\left|(s_n)^2 - s^2\right| &= |s_n - s|\,|s_n + s| \\ &\leq |s_n - s|\,(|s_n| + |s|) \\ &\leq |s_n - s|\,(M + |s|) \\ &< \epsilon\end{aligned}$$

Which gives:

$$|s_n - s| < \frac{\epsilon}{M + |s|}$$

STEP 2: Actual Proof

First of all, since (s_n) converges, (s_n) is bounded, so there is $M > 0$ such that $|s_n| \leq M$ for all n.

Let $\epsilon > 0$ be given

Then since $s_n \to s$ there is N such that for all $n > N$, $|s_n - s| < \frac{\epsilon}{M+|s|}$

With that same N, if $n > N$, we get:

$$\begin{aligned}\left|(s_n)^2 - s^2\right| &= |s_n - s|\,|s_n + s| \\ &\leq |s_n - s|\,(|s_n| + |s|) \\ &\leq |s_n - s|\,(M + |s|) \\ &< \left(\frac{\epsilon}{M + |s|}\right)(M + |s|) \\ &= \epsilon \checkmark\end{aligned}$$

Therefore $(s_n)^2$ converges to s^2 $\square$

3.

Suppose by contradiction that $\sup(B) = M$ where $M < \infty$. Since B has at least one positive term, we may assume $M > 0$

Now consider $M_1 = \frac{M}{2} < M$ (since $M > 0$). By definition of sup this means there is $2^n \in B$ such that $2^n > \frac{M}{2}$, which implies $M < 2^{n+1}$.

But this contradicts the fact that M is an upper bound for B, so *all* $n \in \mathbb{N}$, $2^n \leq M \Rightarrow\Leftarrow$ $\square$

4.

Scratchwork: Notice that $3 = 1 + 2$, so by the binomial theorem, we get:

$$\begin{aligned} 3^n =& (1+2)^n \\ =& 1^n + n1^{n-1}2 + \text{ POSITIVE JUNK} \\ =& 1 + 2n + \text{ POSITIVE JUNK} \\ >& 2n \\ >& M \end{aligned}$$

Which suggests $N = \frac{M}{2}$.

Actual Proof: Let $M > 0$ be given and let $N = \frac{M}{2}$. Then if $n > N$, we have:

$$\begin{aligned} 3^n =& (1+2)^n \\ =& 1 + 2n + \text{ POSITIVE JUNK} \\ >& 2n \\ >& 2\left(\frac{M}{2}\right) \\ =& M \checkmark \end{aligned}$$

Therefore $\lim_{n\to\infty} 3^n = \infty$