MATH 140A – MOCK MIDTERM 2 – SOLUTIONS

1. **STEP 1** By assumption with $r = 1$, we know that $(s-1, s+1)$ has infinitely many terms of (s_n), hence it must include at least one term, let's call it s_{n_1} Hence $s_{n_1} \in (s-1, s+1)$

   STEP 2 Suppose you defined $s_{n_1}, s_{n_2}, \ldots, s_{n_k}$ with $n_1 < n_2 < \cdots < n_k$ such that $s_{n_j} \in \left(s - \frac{1}{j}, s + \frac{1}{j}\right)$ for all $j = 1, \ldots, k$

   By assumption with $r = \frac{1}{k+1}$, $\left(s - \frac{1}{k+1}, s + \frac{1}{k+1}\right)$ has infinitely many terms of (s_n), hence it must include at least one term different from $s_1, s_2, \ldots, s_{n_k}$. Let's call that term $s_{n_{k+1}}$. So $s_{n_{k+1}} \in \left(r - \frac{1}{k+1}, r + \frac{1}{k+1}\right)$ and $n_{k+1} > n_k$ because otherwise $s_{n_{k+1}}$ would be one of the terms $s_1, s_2, \ldots, s_{n_k}$ $\Rightarrow\Leftarrow$ ✓

   STEP 3 Therefore we have constructed a subsequence (s_{n_k}) such that $s_{n_k} \in \left(s - \frac{1}{k}, s + \frac{1}{k}\right)$ for all k, so $|s_{n_k} - s| < \frac{1}{k}$.

   Claim: $s_{n_k} \to s$

   Let $\epsilon > 0$ be given, let $K = \frac{1}{\epsilon}$, then if $k > K$, we have:

$$|s_{n_k} - s| < \frac{1}{k} < \frac{1}{K} = \frac{1}{\left(\frac{1}{\epsilon}\right)} = \epsilon \quad \square$$

Date: Monday, May 18, 2020.

2. **STEP 1:** Let $\epsilon > 0$ be such that $\alpha < \alpha + \epsilon < 1$. Then, by assumption

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{N\to\infty} \sup\left\{|a_n|^{\frac{1}{n}} \mid n > N\right\} = \alpha$$

By definition of a limit, this means that there is N_1 such that if $N > N_1$, then

$$\left|\sup\left\{|a_n|^{\frac{1}{n}} \mid n > N\right\} - \alpha\right| < \epsilon \Rightarrow \sup\left\{|a_n|^{\frac{1}{n}} \mid n > N\right\} - \alpha < \epsilon$$
$$\Rightarrow \sup\left\{|a_n|^{\frac{1}{n}} \mid n > N\right\} < \alpha + \epsilon$$

Since this is true for all $N > N_1$, it is in particular true for *some* N. By definition of sup, this means that for all $n > N$, we have: $|a_n|^{\frac{1}{n}} < \alpha + \epsilon$

STEP 2: But then, if $n > N$, we have: $|a_n| < (\alpha + \epsilon)^n$

Therefore

$$\sum_{n=N+1}^{\infty} |a_n| \leq \sum_{n=N+1}^{\infty} (\alpha + \epsilon)^n = \sum_{n=N+1}^{\infty} r^n$$

But the latter is just a geometric series with $r = \alpha + \epsilon < 1$, so converges, and therefore, by the comparison test, $\sum_{n=N+1}^{\infty} |a_n|$ converges, and so

$$\sum |a_n| = |a_1| + |a_2| + \cdots + |a_N| + \sum_{n=N+1}^{\infty} |a_n| \text{ converges}$$

Hence $\sum |a_n|$ converges, so $\sum a_n$ converges absolutely. □

3. **STEP 1:** Let $(x^{(n)}) = (x_1^n, x_2^n)$ be a Cauchy sequence in $\mathbb{R}^2$. Then for all $\epsilon > 0$ there is N such that if $m, n > N$

$$d\left(x^{(n)}, x^{(m)}\right) = |x_1^n - x_1^m| + |x_2^n - x_2^n| < \epsilon$$

But then, in particular, if $m, n > N$, then

$$|x_1^n - x_1^m| \leq |x_1^n - x_1^m| + |x_2^n - x_2^n| < \epsilon$$

And

$$|x_2^n - x_2^m| \leq |x_1^n - x_1^m| + |x_2^n - x_2^n| < \epsilon$$

So (x_1^n) and (x_2^n) are Cauchy sequences in $\mathbb{R}$

STEP 2: Therefore, since $\mathbb{R}$ is complete, (x_1^n) and (x_2^n) converge to some x_1 and x_2 in $\mathbb{R}$. Let $x = (x_1, x_2)$

STEP 3: **Claim:** $(x^{(n)})$ converges to x.

Let $\epsilon > 0$ be given, then since $x_1^n \to x_1$ there is N_1 such that if $n > N_1$, then $|x_1^n - x_1| < \frac{\epsilon}{2}$

And since $x_2^{(n)} \to x_2$ there is N_2 such that if $n > N_2$, then $|x_2^n - x_2| < \frac{\epsilon}{2}$

Now let $N = \max\{N_1, N_2\}$, then if $n > N$, we get:

$$d\left(x^{(n)}, x\right) = |x_1^n - x_1| + |x_2^n - x_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\checkmark \quad \square$$

4. **STEP 1:** Let's show $\overline{E^c} \subseteq (E^\circ)^c$

Suppose $x \in \overline{E^c}$, then there is a sequence (s_n) in E^c that converges to x. Since $s_n \in E^c$, we must have $s_n \notin E$

Since $s_n \to x$, for all $r > 0$ there is N such that if $n > N$, then $d(s_n, x) < r \Rightarrow s_n \in B(x, r)$

But then this means that for all $r > 0$, $B(x, r) \not\subseteq E$ because $B(x, r)$ contains at least one element s_n that is not in E.

So by definition of E°, $x \notin E^\circ$, so $x \in (E^\circ)^c$ ✓

STEP 2: Let's show $(E^\circ)^c \subseteq \overline{E^c}$.

Suppose $x \in (E^\circ)^c$, so $x \notin E^\circ$, and therefore for every $r > 0$, $B(x, r) \not\subseteq E$, which means for every $r > 0$, $B(x, r)$ must contain at least one element not in E.

But with $r = \frac{1}{n}$, this means that $B\left(x, \frac{1}{n}\right)$ must contain at least one element, call it s_n in E^c.

Claim: $s_n \to x$

Let $\epsilon > 0$ be given, let $N = \frac{1}{\epsilon}$, then if $n > N$, we have

$$d(s_n, x) < \frac{1}{n} < \frac{1}{N} = \frac{1}{\frac{1}{\epsilon}} = \epsilon$$

(the first inequality follows from $s_n \in B\left(x, \frac{1}{n}\right)$)

But then, since (s_n) is a sequence in E^c that converges to x, we get $x \in \overline{E^c}$ ✓ □

4. (a) Let E and F be compact subsets of S, and suppose $\mathcal{U}$ is an open cover of $E \cup F$. We want to show that $\mathcal{U}$ has a finite sub-cover.

But since $E \subseteq E \cup F$, $\mathcal{U}$ covers E. Hence, since E is compact, there is a finite sub-cover $\mathcal{V}_1 = \left\{U_1^1, \dots, U_{N_1}^1\right\}$ of $\mathcal{U}$

Similarly, since $F \subseteq E \cup F$, $\mathcal{U}$ covers F. Hence, since F is compact, there is a finite sub-cover $\mathcal{V}_2 = \left\{U_1^2, \dots, U_{N_2}^2\right\}$ of $\mathcal{U}$

Let $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, then $\mathcal{V}$ is a subset of $\mathcal{U}$ that has at most $N_1 + N_2$ elements, hence is finite. Moreover, $\mathcal{V}$ covers $E \cup F$, because if $x \in E \cup F$, then either $x \in E$, so there is $U_k^1 \in \mathcal{V}$ with $x \in U_k^1$, or $x \in F$, so there is $U_k^2 \in \mathcal{V}$ with $x \in U_k^2$.

Hence $\mathcal{V}$ is a finite sub-cover of $\mathcal{U}$, and therefore $\mathcal{U}$ has a finite sub-cover. Hence $E \cup F$ is compact $\square$

(b) Suppose E and F are compact subsets of S, then in particular E and F are closed. But then $E \cup F$ is closed, so $E \cup F$ is a closed subset of a compact set E, and hence $E \cap F$ is compact.