MORE TOPOLOGY

In this set of notes, we will explore another fascinating facet of topology, namely continuity and connectedness.

1. Continuity in Metric Spaces

Video: Metric Space Continuity

The definition of continuity can be generalized to metric spaces

Definition:

If (S, d) and (S', d') are metric spaces with $f : S \to S'$

Then f is **continuous** at $x_0 \in S$ if for all $\epsilon > 0$ there is $\delta > 0$ such that for all x,

$$d(x, x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \epsilon$$

f is **continuous** if f is continuous at x_0 for all $x_0 \in S$

Problem 1: Let (S, d) be any metric space, and consider $(\mathbb{R}^k, d')$ where d' is the usual metric:

$$d'((x_1, \dots, x_k), (y_1, \dots, y_k)) = \sqrt{\sum_{j=1}^{k} (y_j - x_j)^2}$$

Show that $f = (f_1, \dots, f_k) : S \to \mathbb{R}^k$ is continuous if and only if each component $f_j : S \to \mathbb{R}$ is continuous (where $\mathbb{R}$ is equipped with the usual metric).

Solution: ($\Rightarrow$) Let $\epsilon > 0$ be given, then there is $\delta > 0$ such that if $d(x, x_0) < \delta$, then $d'(f(x), f(x_0)) < \epsilon$.

But, with that same δ, if $d(x, x_0) < \delta$, then for each j,

$$|f_j(x) - f_j(x_0)| = \sqrt{(f_j(x) - f_j(x_0))^2} \leq \sqrt{\sum_{j=1}^{k} (f_j(x) - f_j(x_0))^2} < \epsilon \checkmark$$

Hence f_j is continuous.

($\Leftarrow$) Let $\epsilon > 0$ be given, then for each j, there is $\delta_j > 0$ such that if $d(x, x_0) < \delta_j$, then $|f_j(x) - f_j(x_0)| < \frac{\epsilon}{\sqrt{k}}$

Let $\delta = \min\{\delta_1, \dots, \delta_k\} > 0$, then if $d(x, x_0) < \delta$, then

$$d(f(x), f(x_0)) = \sqrt{\sum_{j=1}^{k} (f_j(x) - f_j(x_0))} < \sqrt{\sum_{j=1}^{k} \left(\frac{\epsilon}{\sqrt{k}}\right)^2} = \sqrt{\sum_{j=1}^{k} \frac{\epsilon^2}{k}}$$
$$= \sqrt{k\left(\frac{\epsilon^2}{k}\right)} = \sqrt{\epsilon^2} = \epsilon \checkmark$$

Hence f is continuous □

Problem 2: Let (S, d) be $\mathbb{R}$ equipped with the discrete metric

$$d(x, y) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases}$$

And let (S', d') be any metric space. Show that *any* function $f : S \to S'$ must be continuous

Video: Every function is continuous

Solution: Let $\epsilon > 0$ be given, let $\delta = \frac{1}{2}$, then if $d(x, x_0) < \delta = \frac{1}{2} < 1$, then $x = x_0$, and therefore

$$d'(f(x), f(x_0)) = d'(f(x_0), f(x_0)) = 0 < \epsilon\checkmark$$

Hence any f is continuous $\square$

Problem 3: This problem is taken from the Berkeley Pre-lim, which is an exam given to first year graduate students at Berkeley, and is therefore quite challenging ☺

Suppose that $f : \mathbb{R}^k \to \mathbb{R}$ (with their usual metrics) satisfies the following two conditions:

(1) For each compact set K, $f(K)$ is compact

(2) For any nested decreasing sequence of compact sets $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$, we have

$$f\left(\bigcap K_n\right) = \bigcap f(K_n)$$

Show that f is continuous

Video: Berkeley Prelim Problem

Solution: **STEP 1:** Fix $x_0 \in \mathbb{R}^k$ and let $\epsilon > 0$ be given. Let $K_n = \overline{B(x_0, \frac{1}{n})}$, notice that the K_n are decreasing, and therefore, by (2), we have

$$\bigcap_{n=1}^{\infty} f(K_n) = f\left(\bigcap_{n=1}^{\infty} K_n\right) = f(\{x_0\}) = \{f(x_0)\}$$

STEP 2: Let $B = B(f(x_0), \epsilon) = (f(x_0) - \epsilon, f(x_0) + \epsilon)$.

Then, first of all

$$\bigcap (f(K_n) \backslash B) = \left(\bigcap f(K_n)\right) \cap B^c = \{f(x_0)\} \backslash B = \emptyset$$

(because $f(x_0)$ is in B)

On the other hand, since K_n is compact, by (1), $f(K_n)$ is compact and hence closed, and so $f(K_n) \backslash B = f(K_n) \cap B^c$ is closed. And since the K_n are decreasing, the $f(K_n)$ are decreasing, and so is $f(K_n) \backslash B$.

Now if for all n, $(f(K_n) \backslash B) \neq \emptyset$, then by the finite intersection property we would have $\bigcap (f(K_n) \backslash B) \neq \emptyset$, which contradicts the above.

Therefore, for some N, $f(K_N) \backslash B = f(K_n) \cap B^c = \emptyset$.

STEP 3: But this implies that $f(K_N) \subseteq B$, and therefore, if $|x - x_0| < \frac{1}{N} \leq \frac{1}{N}$, then $x \in \overline{B(x_0, \frac{1}{N})} = K_N$, and so $f(x) \in f(K_N) \subseteq B = B(f(x_0), \epsilon)$, meaning $|f(x) - f(x_0)| < \epsilon$. In other words

$$|x - x_0| < \frac{1}{N} \Rightarrow |f(x) - f(x_0)| < \epsilon$$

STEP 4: Now given $\epsilon > 0$, let $\delta < \frac{1}{N}$ as above, then if $|x - x_0| < \delta < \frac{1}{N}$, then $|f(x) - f(x_0)| < \epsilon$, and therefore f is continuous at x_0, and hence is continuous. □

2. Continuity in Topology

Video: Topological Continuity

There is a way of talking about continuity without mentioning $\epsilon - \delta$ or sequences at all. This is the one commonly used in topology:

Definition:

If $f : \mathbb{R} \to \mathbb{R}$, and U is any subset of $\mathbb{R}$, then the **pre-image** $f^{-1}(U)$ is defined by

$$x \in f^{-1}(U) \Leftrightarrow f(x) \in U$$

Note: The above definition works for *any* function f, not just invertible ones!

Example: $f(x) = 2x + 3$, then $f^{-1}((5, 9)) = (1, 3)$ because

$$\begin{aligned} x \in f^{-1}((5,9)) \Leftrightarrow & f(x) \in (5,9) \\ \Leftrightarrow & 5 < 2x + 3 < 9 \\ \Leftrightarrow & 2 < 2x < 6 \\ \Leftrightarrow & 1 < x < 3 \end{aligned}$$

Problem 4: Calculate $f^{-1}(U)$ for the following functions f and the following sets U

(a) $f(x) = 3x + 7$, $U = (7, 10)$

(b) $f(x) = x^2$, $U = (-1, 4)$

(c) $f(x) = \sin(x)$, $U = (0, 1)$

Note: Observe that in all of the examples, both U and $f^{-1}(U)$ are open! This is precisely because f is continuous (in topology, this is taken as the *definition* of continuity, since it only involves open sets)

Solution:

(a)

$$\begin{aligned} x \in f^{-1}((7,10)) &\Leftrightarrow f(x) \in (7,10) \\ &\Leftrightarrow 7 < 3x + 7 < 10 \\ &\Leftrightarrow 0 < 3x < 3 \\ &\Leftrightarrow 0 < x < 1 \end{aligned}$$

Hence $f^{-1}(U) = (0, 1)$

(b)

$$\begin{aligned} x \in f^{-1}((-1,4)) &\Leftrightarrow f(x) \in (-1,4) \\ &\Leftrightarrow -1 < x^2 < 4 \\ &\Leftrightarrow -2 < x < 2 \end{aligned}$$

Hence $f^{-1}(U) = (-2, 2)$

(c)

$$\begin{aligned} x \in f^{-1}((0,1)) &\Leftrightarrow f(x) \in (0,1) \\ &\Leftrightarrow 0 < \sin(x) < 1 \\ &\Leftrightarrow x \in \left(2\pi m, 2\pi m + \frac{\pi}{2}\right) \cup \left(2\pi m + \frac{\pi}{2}, (2m+1)\pi\right), m \in \mathbb{Z} \end{aligned}$$

Hence

$$f^{-1}((0,1)) = \bigcup_{m \in \mathbb{Z}} \left(2\pi m, 2\pi m + \frac{\pi}{2}\right) \cup \left(2\pi m + \frac{\pi}{2}, (2m+1)\pi\right)$$

Fact:

$f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if

$$U \text{ is open } \Rightarrow f^{-1}(U) \text{ is open}$$

Problem 5: Prove this fact

Solution: ($\Rightarrow$) Suppose f is continuous and let U be open. We want to show $f^{-1}(U)$ is open.

Let $x_0 \in f^{-1}(U)$. Then, by definition $f(x_0) \in U$. Since U is open, there is $\epsilon > 0$ such that $(f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq U$

However, since f is continuous, there is $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

Claim: $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(U)$

(Then we're done because this shows $f^{-1}(U)$ is open)

Suppose $x \in (x_0 - \delta, x_0 + \delta)$, then $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq U$, and so $f(x) \in U$ and so $x \in f^{-1}(U)$ ✓

($\Leftarrow$) Suppose $f^{-1}(U)$ is open whenever U is open, and let's show f is continuous.

Fix x_0 Let $\epsilon > 0$ be given, then notice that $U = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ is open, and therefore, by assumption, $f^{-1}(U)$ is open.

Moreover, since $f(x_0) \in U$, $x_0 \in f^{-1}(U)$ (which is open), and therefore, by definition, there is $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(U)$

But then, with that δ if $|x - x_0| < \delta$, then $x \in (x_0 - \delta, x_0 + \delta)$ and so $x \in f^{-1}(U)$, which means $f(x) \in U = (f(x_0) - \epsilon, f(x_0) + \epsilon)$, so $|f(x) - f(x_0)| < \epsilon$, and so f is continuous at x_0, and hence continuous ✓ □

Problem 6: To illustrate the elegance of the above definition, let's give a quick proof of the fact that composition of continuous functions are continuous

(a) If f and g are any functions (not necessarily invertible), prove that
$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

(b) Use (a) and the definition above to show that if f and g are continuous, then $g \circ f$ is continuous

Solution:

(a)
$$\begin{aligned} x \in (g \circ f)^{-1}(U) &\Leftrightarrow (g \circ f)(x) \in U \\ &\Leftrightarrow g(f(x)) \in U \\ &\Leftrightarrow f(x) \in g^{-1}(U) \\ &\Leftrightarrow x \in f^{-1}\left(g^{-1}(U)\right) \end{aligned}$$

(b) Suppose U is open, then since g is continuous, $g^{-1}(U)$ is open, and hence, since f is continuous, $f^{-1}\left(g^{-1}(U)\right)$ is open, and therefore

$$(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right) \text{ is open } \checkmark$$

Hence $g \circ f$ is continuous $\square$

Problem 7: Prove that, for any function f and any sets A and B, we have

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(b) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

(c) $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$

Solution:

(a)

$$\begin{aligned} x \in f^{-1}(A \cup B) &\Leftrightarrow f(x) \in A \cup B \\ &\Leftrightarrow (f(x) \in A) \text{ or } (f(x) \in B) \\ &\Leftrightarrow \left(x \in f^{-1}(A)\right) \text{ or } \left(x \in f^{-1}(B)\right) \\ &\Leftrightarrow x \in f^{-1}(A) \cup f^{-1}(B) \end{aligned}$$

(b)

$$\begin{aligned} x \in f^{-1}(A \cap B) &\Leftrightarrow f(x) \in A \cap B \\ &\Leftrightarrow (f(x) \in A) \text{ and } (f(x) \in B) \\ &\Leftrightarrow \left(x \in f^{-1}(A)\right) \text{ and } \left(x \in f^{-1}(B)\right) \\ &\Leftrightarrow x \in f^{-1}(A) \cap f^{-1}(B) \end{aligned}$$

(c)

$$\begin{aligned} x \in f^{-1}(A^c) &\Leftrightarrow f(x) \in A^c \\ &\Leftrightarrow f(x) \notin A \\ &\Leftrightarrow \text{ Not } \ (f(x) \in A) \\ &\Leftrightarrow \text{ Not } \ \left(x \in f^{-1}(A)\right) \\ &\Leftrightarrow x \notin f^{-1}(A) \\ &\Leftrightarrow x \in \left(f^{-1}(A)\right)^c \end{aligned}$$

Definition:

Given a function f and a subset A of $\mathbb{R}$, we define

$$f(A) = \{f(x) \mid x \in A\}$$

Problem 8: Here's a nice exercise using compactness and pre-images

(a) Show that if K is (covering) compact and f is continuous, then $f(K)$ is (compact)

(b) Is there a continuous function f with domain $[0,1]$ and range $(0,1)$?

(c) Show that any continuous function from $[a,b]$ to $\mathbb{R}$ must be bounded

Video: Continuity and Compactness

Solutions:

(a) **STEP 1:** Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of $f(K)$, and consider $\mathcal{U}' = \left\{f^{-1}(U_\alpha)\right\}$.

STEP 2: Then, since U_α is open and f is continuous, $f^{-1}(U_\alpha)$ is open.

Moreover, by an analog of the above problem, we have

$$\bigcup_\alpha f^{-1}(U_\alpha) = f^{-1}\left(\bigcup_\alpha U_\alpha\right)$$

And, since $\mathcal{U}$ covers $f(K)$, we have $K \subseteq \bigcup_\alpha U_\alpha$ and so $f^{-1}\left(\bigcup_\alpha U_\alpha\right) \supseteq f^{-1}(f(K))$

And finally $K \subseteq f^{-1}(f(K))$ since if $x \in K$, then $f(x) \in f(K)$ and so $x \in f^{-1}(f(K))$.

Therefore, combining everything, we get

$$\bigcup_\alpha f^{-1}(U_\alpha) \supset K$$

STEP 3: So $\mathcal{U}'$ covers K. But since K is compact, there is a finite sub-cover

$$\mathcal{V}' = \left\{f^{-1}(U_{n_1}), \dots, f^{-1}(U_{n_N})\right\}$$

STEP 4:

Claim:

$$\mathcal{V} =: \{U_{n_1}, \dots, U_{n_N}\}$$

Covers K

(Then we're done because we found a finite sub-cover of $\mathcal{U}$)

But if $y \in f(K)$, then $y = f(x)$ for some $x \in K$, but since $\mathcal{V}$ covers K, $x \in f^{-1}(U_{n_k})$ for some k, and so $y = f(x) \in U_{n_k} \in \mathcal{V}$ ✓ □

(b) No since $[0,1]$ is compact, and so $f([0,1])$ would be compact, but $f([0,1]) = (0,1)$, which is not compact

(c) Since $[a,b]$ is compact and f is continuous, $f([a,b])$ is compact, and therefore bounded, which means that f is bounded (that is there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$)

Problem 9: Give a quick proof of the Extreme Value Theorem: If K is a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is continuous, then f attains a maximum and a minimum

Solution: Since K is compact, and f is continuous, $f(K)$ is compact by the problem above. Since $f(K)$ is compact, it is closed and bounded, and therefore it has a least upper bound $M = \sup(f(K))$

Let (y_n) be a sequence in $f(K)$ converging to M. By definition of $f(K)$, $y_n = f(x_n)$ for some $x_n \in K$
But since K is (covering) compact, K is sequentially compact, and therefor (x_n) has a convergent subsequence (x_{n_k}) that converges to some $x_0 \in K$

But since f is continuous, we get $f(x_{n_k}) \to f(x_0)$.

But then since y_n converges to M, the subsequence $y_{n_k} = f(x_{n_k})$ converges to M, so by uniqueness of limits, $f(x_0) = M$, so f has a maximum M at $x_0 \in K$, and similarly f has a minimum m at some other point. □

3. Connectedness

Video: Connectedness

Definition:

Let E be any subset of $\mathbb{R}$ (or of any metric space)

(1) E is **disconnected** if there are disjoint, nonempty, and open subsets A and B of E such that $A \cup B = E$

(2) E is **connected** if it is not disconnected

For example, $\mathbb{R}$ is connected but $(0, 1) \cup (2, 3)$ is disconnected

Problem 10: Give a short proof of the Intermediate Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and c is between $f(a)$ and $f(b)$, then there is $x \in [a, b]$ with $f(x) = c$. Isn't connectedness awesome?

Solution: Suppose not, then there is c such that $f(x) \neq c$ for all $x \in [a, b]$. This means that for all x, either $f(x) > c$ or $f(x) < c$, and therefore $[a, b] = A \cup B$ where

$$A = \{x \in [a, b] \mid f(x) < c\} = f^{-1}((-\infty, c))$$
$$B = \{x \in [a, b] \mid f(x) > c\} = f^{-1}((c, \infty))$$

Now $A \cup B = \emptyset$ and A and B are nonempty since either $f(a)$ or $f(b)$ are in A or B

Moreover, A and B are open since f is continuous and $(-\infty, c)$ and (c, ∞) are open.

And therefore $[a, b] = A \cup B$ with A and B nonempty, open, and disjoint, which contradicts the fact that $[a, b]$ is connected. $\Rightarrow\Leftarrow$ $\square$

Problem 11: Suppose E is connected and $f : E \to \mathbb{R}$ is continuous, prove that $f(E)$ is connected.

Suppose E is connected by $f(E)$ is not connected. Then there are A and B nonempty, open, and disjoint with $f(E) = A \cup B$.

But now consider $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$. Then, since A and B are open and f is continuous, we get A' and B' are open. Moreover:

$$A' \cap B' = f^{-1}(A) \cap f^{-1}(B) = f^{-1}\left(A \cap B\right) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$$

$$A' \cup B' = f^{-1}(A) \cup f^{-1}(B) = f^{-1}\left(A \cup B\right) = f^{-1}(f(E)) = E$$

(The latter follows because for all $x \in E$, $f(x) \in f(E)$ and therefore $x \in f^{-1}\left(f(E)\right)$)

Finally, since A is nonempty there is $a \in A \subseteq f(E)$ and therefore there is $a' \in E$ with $f(a') \in A$ and so $a' \in f^{-1}(A) = A'$ and so A' is nonempty, and similarly B' is nonempty.

Therefore A' and B' are disjoint, nonempty, and open subsets of E with $A' \cup B' = E$, but this implies that E is disconnected $\Rightarrow\Leftarrow$

Problem 12: Prove that $\mathbb{R}$ is connected. More generally, it follows that any interval I is connected.

Video: $\mathbb{R}$ is connected

Solution: Suppose $\mathbb{R}$ is not connected. Then we can write $\mathbb{R} = A \cup B$ with A, B nonempty, open and disjoint.

STEP 1: Since A and B are nonempty, fix $a \in A$ and $b \in B$. WLOG $a < b$ ($a \neq b$ since A and B are disjoint) and consider

$$S = \{x \in [a, b] \mid [a, x] \subseteq A\}$$

Then S is nonempty since $a \in S$ and moreover S is bounded above by b, hence S has a least upper bound $M = \sup(S)$

STEP 2:

Claim: $M \notin B$

Suppose $M \in B$. Then since B is open, there is $r > 0$ such that $(M - r, M + r) \subseteq B$.

Since $M - r < M = \sup(S)$, there is $x \in S$ such that $x > M - r$. Since $x \in S$, we get $[a, x] \subseteq A$, and so $x \in A$. But, on the other hand $x \in (M - r, M] \subseteq (M - r, M + r) \subseteq B$, and therefore $x \in A \cap B = \emptyset \Rightarrow\Leftarrow$. Hence, since $M \notin B$ and $A \cup B = \mathbb{R}$, we must have $M \in A$

STEP 3: Moreover $M \in S$, because if $M \notin S$, then $[a, M] \nsubseteq A$, meaning there is $x \in [a, M]$ with $x \notin A$. But since $M \in A$, we have $x < M = \sup(S)$ and therefore there is $y \in S$ with $y > x$. But by definition of S, we have $[a, y] \subseteq A$ and so, since $x < y$ we get $[a, x] \subseteq [a, y] \subseteq A$, which is a contradiction since $x \notin A$.

STEP 4: Now $M < b$, because if $b \leq M$, then we get a contradiction because, since $M \in S$, we have $[a, M] \subseteq A$ and so $b \in [a, M] \subseteq A$ so

$b \in A \Rightarrow\Leftarrow$

STEP 5:

Claim: $M \notin A$

Suppose $M \in A$, then, since A is open, there is $r' > 0$ such that $(M - r', M + r') \subseteq A$. Let $M' = \min\{M + r', b\}$

Then $M' > M$, and so $M' \notin S$ because $M = \sup(S)$.

Therefore, by definition of S, $[a, M'] \not\subseteq A$, so there is some $x \in [a, M']$ with $x \notin A$. But since $[a, M] \subseteq A$ (because $M \in S$), we must have $x \in (M, M']$. Moreover, $x \neq M + r'$ (because $M + r' \in A$ but $x \notin A$), and therefore $x \in (M, M + r') \subseteq A$, so $x \in A \Rightarrow\Leftarrow$.

Hence $M \notin A$ either, and therefore M is neither in A or in B, which contradicts $\mathbb{R} = A \cup B \Rightarrow\Leftarrow$. □

Definition:

Let E be any subset of $\mathbb{R}$ (or of any metric space)

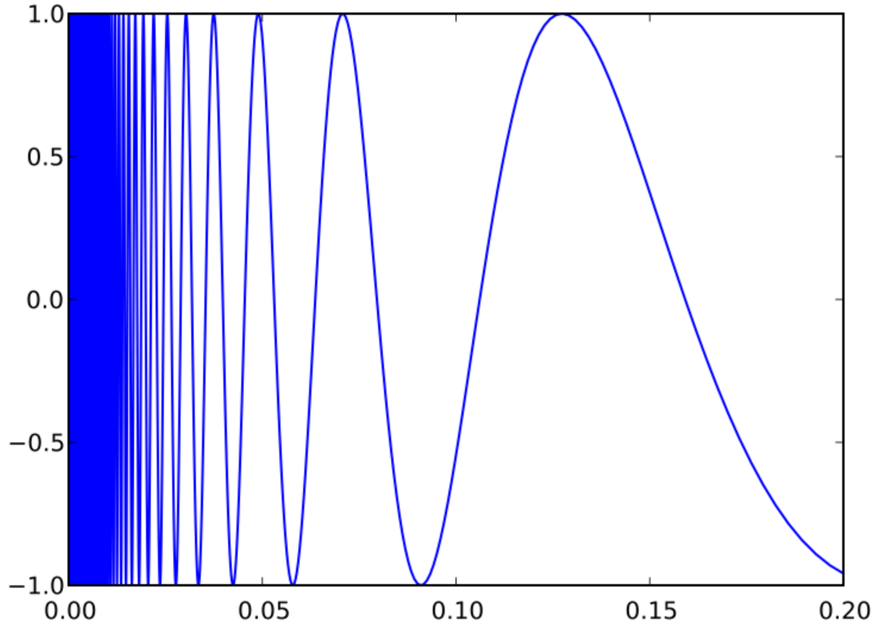
(1) A **path** in E is a continuous function $\gamma : [0, 1] \to E$

(2) E is **path-connected** if for any pair of points a and b in E, there is a path γ with $\gamma(0) = a$ and $\gamma(1) = b$

Problem 13:

(a) Show that if E is path-connected, then it is connected

(b) Show $\mathbb{R}$ is path-connected and deduce that it is connected.



Solution: For (a), suppose E is path-connected but not connected. Since E is not connected, there are A and B, nonempty, open, and disjoint such that $A \cup B = E$.

Since A and B are nonempty, there is $a \in A$ and $b \in B$.

Since γ is path-connected, there is a path $\gamma : [0, 1] \to E$ with $\gamma(0) = a$ and $\gamma(1) = b$

Now consider $A' = \gamma^{-1}(A)$ and $B' = \gamma^{-1}(B)$. Then since A and B are open and γ is continuous, we get A' and B' are open.

Moreover $0 \in A'$ since $\gamma(0) = a \in A$ and therefore A' is nonempty, and similarly B' is nonempty, and finally

$$A' \cap B' = \gamma^{-1}(A' \cap B') = \gamma^{-1}(A') \cap \gamma^{-1}(B') = A \cap B = \emptyset$$
$$A' \cup B' = \gamma^{-1}(A' \cup B') = \gamma^{-1}(A') \cup \gamma^{-1}(B') = A \cup B = [0,1]$$

But therefore A' and B' are disjoint, open, nonempty subsets of $[0,1]$ whose union in $[0,1]$, which contradicts that $[0,1]$ is connected $\Rightarrow\Leftarrow$.

Hence E must be connected

For (b), let $a, b \in \mathbb{R}$ and consider the path $\gamma(t) = (1-t)a + tb$, which is continuous and has values in $\mathbb{R}$ and $\gamma(0) = a$ and $\gamma(1) = b$ ✓

Problem 14: The topologist's sine curve is defined as

$$E = F \cup G =: \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x \in (0,1] \right\} \cup \{\{0\} \times [-1,1]\}$$

Show that E is connected but not path-connected.

Video: Topologist Sine Curve

Solution: **Note:** The solutions here are taken from this handout

Proof that E is connected:

Claim: If F is connected subset of $\mathbb{R}^2$, then $\overline{F}$ is connected

Proof: The result is true of $F = \emptyset$, so assume $F \neq \emptyset$.

Suppose F is connected but $\overline{F}$ is not connected. Then there are open nonempty disjoint subsets A and B of $\overline{F}$ such that $A \cup B = \overline{F}$.

Consider $A' = A \cap F$ and $B' = B \cap F$. Then A' and B' are open in F, disjoint, and their union is F. But since F is connected, we must have $A' = F$ and $B' = \emptyset$ or $A' = \emptyset$ and $B' = F$.

WLOG, assume $A' = F$ and $B' = \emptyset$

Notice that, since $A^c = B$ is open (the complement here is in $\overline{F}$) we get A is closed in $\overline{F}$, hence there is some closed subset C of $\mathbb{R}^2$ with $A = C \cap \overline{F}$

But then $F = A' \subseteq A \subseteq C$, and since $\overline{F}$ is the smallest closed subset containing F, we get $\overline{F} \subseteq C$, and hence

$$A = C \cap \overline{F} = \overline{F}$$

And since A is disjoint from B, we must get that $B = \emptyset \Rightarrow\Leftarrow$. $\square$

To show that our topologist sine curve is connected, let

$$F = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x \in (0,1] \right\}$$

Then F is connected since it is path-connected (for a and b in $(0,1]$, just consider the path $\gamma(t) = \left((1-t)a + tb, \sin\left(\frac{1}{(1-t)a+tb}\right)\right)$), and moreover $\overline{F} = E$ (since $\sin\left(\frac{1}{x}\right)$ has the intermediate value property on $\mathbb{R}$), and therefore by the Claim, E is connected

Proof that E is not path-connected: Suppose not, then in particular is there is $\gamma : [0,1] \to E$ with $\gamma(0) \in F$ and $\gamma(1) \in G$.

Because G is just a straight line (which is path-connected), we may assume $\gamma(1) = (0, 1)$.

Let $\epsilon = \frac{1}{2}$, then by continuity of γ at 1, there is $\delta > 0$ such that

$$\text{If } |t - 1| \leq \delta \Rightarrow 1 - \delta \leq t \leq 1, \text{ then}$$

$$|\gamma(t) - \gamma(1)| < \frac{1}{2} \Rightarrow |\gamma(t) - (0, 1)| < \frac{1}{2}$$

(Note: Here the absolute value for γ is just the usual distance in $\mathbb{R}^2$. Also the $\leq \delta$ isn't really a problem)

Let $\gamma(1 - \delta) =: (x_0, y_0)$ and remember that $\gamma(1) = (0, 1)$

Since $\gamma = (\gamma_1, \gamma_2)$ is continuous, the first component γ_1 is continuous, and therefore, by the Intermediate Value Theorem, γ_1 attains all the values between $\gamma_1(1 - \delta) = x_0$ and $\gamma_1(1) = 0$, and hence $\gamma_1([1 - \delta, 1])$ contains the interval $[0, x_0]$

Hence for all $x_1 \in (0, x_0]$ there is some t with $\gamma_1(t) = x_1$ and therefore, by definition, there is $t \in [1 - \gamma, 1]$ such that

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) = \left(x_1, \sin\left(\frac{1}{x_1}\right)\right)$$

But now let $x_1 = \frac{1}{2\pi n - \frac{\pi}{2}}$, then for n large enough we have $0 < x_1 < x_0$, but $\sin\left(\frac{1}{x_1}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$
Hence the point $\left(\frac{1}{2\pi n - \frac{\pi}{2}}, -1\right)$ has the form $f(t)$ for some $t \in [1 - \delta, 1]$ and hence t is a distance of $\frac{1}{2}$ away from $(0, 1)$, which contradicts the fact that the distance between $\left(\frac{1}{2\pi n - \frac{\pi}{2}}, -1\right)$ and $(0, 1)$ is at least 2

$\Rightarrow\Leftarrow$ □

4. Homeomorphisms

Video: Homeomorphism

Definition:

Let A and B be two subsets of $\mathbb{R}$ (or any two metric spaces) and $f : A \to B$ is a function, then:

(a) f is a **homeomorphism** if f is continuous, one-to-one, onto, and f^{-1} is continuous

(b) A and B are **homeomorphic** if there is a homemorphism between A and B

(c) A **topological property** is a property that is preserved under homeomorphisms

Problem 15:

(a) Show that there is a homeomorphism between $(0,1)$ and $\mathbb{R}$. So surprisingly $(0,1)$ and $\mathbb{R}$ are homeomorphic

(b) Deduce that boundedness is not a topological property.

Solution: For (a), consider $f : (0,1) \to \mathbb{R}$ defined by

$$f(x) = \tan^{-1}\left(\pi x - \frac{\pi}{2}\right)$$

Then, one can check that $g(x) = \pi x - \frac{\pi}{2}$ is continuous, one-to-one, and onto, and its inverse is continuous and therefore a homeomorphism.

Also since $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is continuous and one-to-one and onto $\mathbb{R}$ (you can show this using the fact that $\tan(x) \to \pm\infty$ near $\pm\frac{\pi}{2}$ and an analog of the Intermediate Value Theorem), its inverse $\tan^{-1}$ is continuous, and therefore a homeomorphism

Hence $f(x)$ is a homeomorphism, being a composition of two homeomorphisms, and therefore $(0,1)$ and $\mathbb{R}$ are homeomorphic.

For (b), since $(0,1)$ is bounded but $\mathbb{R}$ is unbounded, boundedness is not a topological property.

Problem 16:

(a) Show that if $f : I \to f(I)$ is continuous and one-to-one, then f is a homeomorphism

(b) Show that if K is covering compact and $f : K \to f(K)$ is continuous and one-to-one, then f is a homeomorphism

(c) Let S^1 be the unit circle in $\mathbb{R}^2$. Consider the map $f : [0, 2\pi) \to S^1$ by $f(t) = (\cos(t), \sin(t))$. You may assume that f is continuous, one-to-one, and onto. Show that f^{-1} is not continuous and hence not a homeomorphism.

Solution:

(a) By assumption, f is continuous, one-to-one, and onto its image $f(I)$. Moreover, we have shown in class that f^{-1} is continuous, hence f is a homeomorphism.

(b) Since f is continuous, one-to-one, and onto its image, it suffices to show that f^{-1} is continuous.

Claim: f is continuous if and only if for each closed set C, $f^{-1}(C)$ is closed

This follows because if f is continuous and C is closed, then C^c is open, and therefore $f^{-1}(C^c)$ is open, hence $\left(f^{-1}(C)\right)^c$ is open, so $f^{-1}(C)$ is closed ✓

Conversely, if $f^{-1}(C)$ is closed whenever C is closed, then if U is any open set, then U^c is closed, so by assumption $f^{-1}(U^c)$ is closed, and therefore $\left(f^{-1}(U)\right)^c$ is closed, and so $f^{-1}(U)$ is open, so f is continuous ✓

Now suppose C is an arbitrary closed subset of K, then since K is compact, C is a closed subset of a compact set, and hence compact. Therefore, since C is compact and f is continuous, $f(C)$ is compact, and hence closed.

Therefore, whenever C is closed, $f(C)$ is closed, and by the claim below, it follows that $\left(f^{-1}\right)^{-1}(C) = f(C)$ is closed, and so f^{-1} is continuous since f was arbitrary

Claim: $\left(f^{-1}\right)^{-1}(C) = f(C)$

Proof:

$$\begin{aligned} x \in \left(f^{-1}\right)^{-1}(C) &\Leftrightarrow f^{-1}(x) \in C \\ &\Leftrightarrow f\left(f^{-1}(x)\right) \in f(C) \\ &\Leftrightarrow x \in f(C) \checkmark \quad \square \end{aligned}$$

(c) Let

$$(x_n) = \left(\cos\left(2\pi - \frac{1}{n}\right), \sin\left(2\pi - \frac{1}{n}\right)\right)$$

Then (x_n) converges to $(1,0)$, but $f^{-1}(x_n) = 2\pi - \frac{1}{n}$ converges to $2\pi \neq f^{-1}((1,0)) = 0$.

Hence f^{-1} is not continuous.

Problem 17:

(a) Show that homeomorphisms map compact sets onto compact sets. Hence compactness is a topological property. Deduce that $[0,1]$ and $\mathbb{R}$ are not homeomorphic

(b) Show that homeomorphisms map connected sets onto connected sets. So connectedness is a topological property. Deduce that $[0,2\pi]$ and the unit circle S^1 in $\mathbb{R}^2$ are not homeomorphic

(c) Show openness and closedness are topological properties. Deduce that $(0,1)$ and $[0,1]$ (considered as subsets of $\mathbb{R}$) are not homeomorphic

Solution:

(a) This just follows because if K is compact and f is continuous, then $f(K)$ is compact. Therefore, since $[0,1]$ is compact but $\mathbb{R}$ is not compact, the two spaces are not homeomorphic.

(b) This just follows because if E is connected and f is continuous, then $f(E)$ is connected.

It is not hard to show that if $f : E \to F$ is a homeomorphism and $x_0 \in E$, then $f : E \backslash \{x_0\} \to F \backslash \{f(x_0)\}$ is also a homeomorphism.

Now if $[0, 2\pi]$ and S^1 were homeomorphic with homeomorphism f, then $[0, 2\pi] \backslash \{1\}$ and $S^1 \backslash \{f(1)\}$ would also be a homeomorphism. But this can't be because $[0, 2\pi] \backslash \{1\} = [0, 1) \cup (1, 2\pi]$ is disconnected, whereas S^1 minus a point is still connected! $\Rightarrow\Leftarrow$

(c) Suppose A is open and f is a homemomorphism, then $f(A) = \left(f^{-1}\right)^{-1}(A)$ is open since f^{-1} is continuous and A is open. Similarly, if B is closed, then $f(B) = \left(f^{-1}\right)^{-1}(B)$ is closed since f^{-1} is continuous and B is closed

Now Since $(0, 1)$ is open in $\mathbb{R}$ and $[0, 1]$ is not open in $\mathbb{R}$, those two cannot be homeomorphic.